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Recurrent Price Index Problems and Some Early German Papers on Index Numbers

Notes on Laspeyres, Paasche, Drobisch, and Lehr

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Summary

With Laspeyres, Paasche and other authors such as Drobisch and Lehr, Germany made quite a promising start in index theory in the last decades of the 19th century. However, it soon lost ground after this period, which is described in this paper. The focus is not on biographies but on controversies where these persons acted as opponents and developed the views for which they are well known. The issues selected are primarily those which are still interesting and controversial today, e.g. the merits and demerits of certain index formulas, the definition and updating of weights, "pure" price comparison vs. chain indices etc. However, in order to aid a better understanding of how Laspeyres etc. arrived at their index formulas and views about the purposes index numbers should serve, some attention is also given to the typical disputes and prejudices of that time (e.g. regarding inflation under the regime of a gold currency).

1 Introduction

It appears attractive to take the occasion of the anniversary of this journal to review some papers that appeared in this journal during the last three or four decades of the 19th century, and which paved the way to modern price index numbers. It is particularly appealing as such a plan leads to names that were later to become famous around the world[1],

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[1] It was anything but certain that names like Laspeyres and Paasche were to become so famous. Given the German literature on index numbers in the first half of the 20th century, it does not seem unlikely that these names would have fallen into oblivion had there not been some English speaking authors, in particular the Americans Walsh and Fisher, who constantly referred to their work (and names). Interestingly even L. v. Bortkiewicz in 1932 did not introduce the names Laspeyres and Paasche (unlike his 1927 article) as authors of his formulas 1 and 2, while he consistently mentioned the names in connection with the other eight formulas he discussed in his paper. It is perhaps also not by

for example Etienne Laspeyres (1834 – 1913) and Herrmann Paasche (1851 – 1925), as well as two probably far less well-known authors, namely Moritz Wilhelm Drobisch (1802 – 1896) and Julius Lehr (1845 – 1894).[2] The life and academic work of these celebrities in index theory, each of them taken in isolation, has already found a number of competent and detailed descriptions and recognitions. So we have for example biographies and appreciations of the academic achievements of Laspeyres (Rinne 1981; Diewert 1987; Roberts 2000), Drobisch (v. Auer 2010), and also much is known about Paasche, who for example once was a vice-president of the German *Reichstag* (parliament).

Our intention is therefore already at the outset quite different, namely to relate these persons to some controversial issues in index numbers in which they acted as opponents. We selected such issues which already occupied a lot of people at that time and which still continue to do so today. After a review of many articles (in this and other journals) and also books on indices of that time (roughly 1860 to 1920), it was clear that there should be enough material to carry out this plan, and that we can indeed see a number of such perennial controversies fought out on the long road to modern index number theory. Such controversies and the persons involved in them are

- the use of the geometric or the arithmetic mean[3] in a problem now known as "low-level aggregation" (or compilation of "elementary" indices); historically this is the "Jevons vs. Laspeyres" case and will be dealt with in Sec. 2;
- the introduction of weights (to account for the relative importance of goods), for example physical weights multiplied with prices to form so-called "unit values" and based on them the "unit value index", which is the case "Drobisch vs. Laspeyres" (see Sec. 3), a case which also gave rise to a claim of authorship (on the part of Drobisch; see Sec. 4);
- the choice between a single (using quantity weights q_0 *or* q_t)[4] and a double weighting system (using both the quantities q_0 *and* q_t as weights); see Sec. 5;
- disputes as to whether an index should be compiled as an average of price relatives (price ratios) or rather represent a ratio of average prices (Sec. 6); and finally
- as a sort of logical continuation of the problem that a constant updating of the q_t's appears desirable, we find the idea of a "chain index", in which the periods 0 (base period) and t (current period) are not compared directly with an index P_{0t} affected by prices and quantities of 0 and t only, but via "chaining" (multiplying) $P_{01}P_{12}\ldots P_{t-1,t}$; see Sec. 7.[5]

coincidence that I had problems with getting some German articles of Laspeyres and Paasche here in Germany. So I owe for example copies of Laspeyres 1875 and Paasche 1878 to Othmar Winkler (Georgetown University, Washington D. C.), and Laspeyres 1883 to Hellen Roberts (University of Illinois, Chicago). I also should express my gratitude to Erwin Diewert. I learned a lot from the historical remarks he made in his e-mails.

2 The fourth memorable author (Lehr), to which Sec. 7 below will be devoted, is also repeatedly represented in this journal (in particular with papers about the then revolutionary ["Austrian school"] concepts of "marginal utility" etc.) He presented his considerations about index numbers, however, in a monograph (Lehr 1885). In addition to the fact that renowned authors published their papers on index numbers in this journal, it is noteworthy that J. Conrad, a former editor of the journal, did a lot to promote index number research (see footnotes 60, 62).

3 This dispute is also summarized in Walsh (1901: 220).

4 As is well known, the formulas of Laspeyres and Paasche differ in this respect.

5 Nothing indicates a difference or even controversy between Laspeyres and Paasche; however, it is justified to speak of "Lehr vs. Paasche", because Lehr, as an early proponent of chain indices, quite

Hence our focus is on recurrent and more or less still relevant index problems rather than biographies. We therefore exclude problems that used to trigger some passionate discussions and had a considerable impact on the study of index numbers at the time under consideration here, but have since lost much or all of their relevance. Many of the early debates can only be understood in the light of the (gold) currency problems of that time. For example, gold currency versus bimetallism proved to be a catalyst for Jevons' interest in index numbers. The relationship between inflation, money and prices was not yet well understood. Laspeyres ran into great difficulties with a distinction between rising prices of commodities ("*Waarenbewerthung*", revaluation of goods) on the one hand and devaluation of money ("*Geldentwerthung*") on the other, because both phenomena were observationally equivalent.[6] It is also noticeable that it was not yet generally accepted that inflation would call for a study of prices (as was Laspeyres' view) rather than for statistics of the "*Zunahme des Metallvorraths*" (increased availability of precious metals).[7] The problem people found at least as intriguing as index numbers was for example whether an increase in prices was primarily caused by gold discoveries, or by higher aspiration levels and consumption standards of the urban working class.[8]

Moreover, not only *what* was discussed in those days may appear strange, but also *how* it was discussed. In order to do justice to authors of that time it should be borne in mind that many now well-established methods to assess index formulas had not yet been developed, or at least were not yet familiar. To assess formulas in term of "axioms" (or "tests") was still uncommon, and *a fortiori* to interpret formulas in terms of utility maximizing by making substitutions in response to changes of relative prices. This was at best alluded to in rather vague verbal statements, but definitely not yet worked out mathematically. Of course mathematics for economists was in general only in its infancy.[9] It was not uncommon to content oneself with numerical examples, elaborated in detail over many pages. Also, lengthy deliberations about the correct definitions of certain concepts and logical conditions required for certain conclusions were common practice. It should also be noticed that academic communication across borders and detailed knowledge of foreign publications were but exceptions. It is well known and reported that e.g. Jevons and Laspeyres were in close contact. Finally, it should be added that in

vehemently criticized Paasche for his vague and slightly inconclusive position concerning the weights. In the view of Lehr, Paasche saw (in Paasche 1874) a need for continuously varying weights q_t but inconsequentially he did not go so far as to account also for all the intermediate periods q_{t-1}, q_{t-2},... as vehemently required by Lehr (1885: 44).

6 As Laspeyres (1864: 82) noted, they are "wearing the same outer garment" (*tragen "dasselbe äussere Gewand"*). Also in the title of Drobisch (1871a), we see the distinction between change of prices (*Veränderungen der Waarenpreise*) and change of the exchange value of money (*Veränderungen des Geldwerths*). As to the "exchange value", Walsh 1901 seems to have succeeded in maximizing confusion with a host of hair-splitting terminological distinctions.

7 The title of Paasche 1878 might be mistaken as a book providing statistics of prices or empirical research with his formula, presented just four years ago, but in fact it is dealing almost exclusively with statistics about stocks and flows of gold and silver. There is no mention given to a "price level" let alone a price index.

8 In this point opinions of Laspeyres and Paasche seemed to differ slightly; cp. Laspeyres (1883: 798).

9 A comparison of some writings of German authors to for example Edgeworth's papers on index theory at that time clearly showed that the Germans began already soon after the time of Laspeyres and Paasche to lag behind and they did so even more pronouncedly some decades later. We could find a great number of articles on index numbers in English and American journals in the first couple of decades of the 20th century, but not many in German journals. Instead there was much in the form of futile sophistic philosophy about money and prices.

what follows the focus is on the formula, as is common in index theory, while many non-formula problems such as the selection of commodities for the price index, the organization of regular price quotations and family budget surveys, or how to make sure that the quality of goods at different points in time is comparable are not discussed here. Laspeyres in particular quite often referred to such aspects of indices.

Finally, it is noteworthy and should be borne in mind that in the time period dealt with in this paper there was hardly any systematic statistical data gathering, and official statistical agencies (on a regional or national level) were not yet established, or at least were quite rare. What is now provided by official statistics was simply unavailable or had to be compiled laboriously on a private initiative. Economists like Laspeyres etc. spent an enormous amount of time and effort on the compilation of statistical figures at the expense of mathematical or conceptual work in statistics.

2 Laspeyres vs. Jevons: arithmetic vs. geometric mean of price relatives

Laspeyres dealt mainly with three problems in two famous contributions to this journal. In his paper of 1864 he discussed:

1. the relationship between the quantity and value of gold on the one hand and the level of prices (of commodities) on the other;
2. whether to use the geometric mean of price relatives as suggested by Jevons, or rather to keep to the arithmetic mean as preferred by Laspeyres and most of the economists of his day; while
3. the choice of suitable quantity weights for prices, intended to indicate their relative importance, may be viewed as a third problem, and is dealt with only in Laspeyres 1871, where he also presented his well-known price index formula. The formula grew out of a controversy with Drobisch to which we will return in Section 3.2.

While the first problem is no longer relevant, the second is still an issue now, and is referred to as "low-level aggregation" of price quotations.[10]

In Laspeyres' day, the formula generally in use was the arithmetic mean of price relatives (price ratios), now known as the index formula of Carli:[11]

$$P_{0t}^{C} = \frac{1}{n}\sum_{i=1}^{n}\frac{p_{it}}{p_{i0}}. \tag{1}$$

[10] Such (unweighted) "elementary indices" serve as building blocks for a second aggregation (this time inclusive of weights) when an index is compiled in two stages, which is common practice in official statistics.

[11] This index is also known as the "Sauerbeck index" (see also Balk (2008: 9). Laspeyres and some other contemporary authors made extensive use of this formula (and also of Sauerbeck's price statistics for the British foreign trade; while Sauerbeck provided data for England, Soetbeer did the same for Germany). It was only in the 20th century (more precisely: owing to Walsh 1901) that it became generally known that the formula originated from Gian Rinaldo Carli (1720 – 1795). Walsh also discovered that Dutot was the author of the index P^{D} (see eq. 3). References to the books of Carli, Dutot and many other authors of the early history of index numbers can be found for example in Diewert 1993.

Jevons by contrast suggested the geometric mean (at that time unusual and unfamiliar):

$$P_{0t}^{J} = \left(\prod_{i=1}^{n} \frac{p_{it}}{p_{i0}}\right)^{1/n}. \tag{2}$$

As mentioned above, the problem of which mean to use when an unweighted (using prices only) or "elementary" price index is to be compiled is still relevant. However, we are nowadays in a better position in that we are used to discussing such problems with reference to "tests" (or "axioms"), like for example the time reversal test,[12] or other axioms which were basically unknown at the time of Jevons and Laspeyres. The way in which arguments were developed and advanced, e.g. by numerical examples, was quite different in those days.

For his contemporaries it was widely accepted (and criticized) that Jevons did not give many reasons for his choice of the geometric mean, and when he faced adverse opinions he did not make the issue any clearer by adding yet another candidate, namely the harmonic mean.[13]

Literature on the issue of geometric vs. arithmetic means abounds, and already did so at the time under consideration here, and space restrictions require us to limit ourselves to only those arguments that were expressly advanced by Laspeyres and Jevons in their controversy.

Laspeyres frankly admitted that he was impressed by Jevons's example[14] according to which a rise in the cocoa price of 100 % ($p_{1t}/p_{10} = 2$) will be neutralized by a drop of the price for cloves by 50 % ($p_{2t}/p_{20} = 0.5$), so that $P^J = 1$ should be correct by contrast to $P^C = 1.25$. In his rebuttal, Laspeyres (arguing again in terms of a numerical example) made the assumption that if we initially have one "Centner", i.e. one hundredweight (1 cwt.) cocoa for $p_{10} = 100$ Thaler (Tlr), and also 1 cwt. clove for $p_{20} = 100$ Tlr, the change in prices means that we later (with new prices $p_{1t} = 200$ and $p_{2t} = 50$) will get only $q_{1t} = 0.75$ cwt. (instead of 1 cwt.) cocoa and $q_{2t} = q_{20} = 1$ cwt. cloves. The 1/4 cwt. cocoa less is worth 50 Tlr (or 25 % of the expenditure $\Sigma p_0 q_0 = \Sigma p_t q_t = 200$). So prices in actual fact rose by 25 % instead of the 0 % according to Jevons, and $P^C = 1.25$ is correct.[15]

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[12] This test requires that interchanging 0 and t in an index should result in $P_{t0} = (P_{0t})^{-1}$. It is in no small measure due to this test that P^C came out on the losing end of the rivalry with P^J. But time reversibility was not yet an issue in the controversy Laspeyres vs. Jevons. The concept of this test is due to Pierson (1896: 128). More formal statements of the test were made by Walsh (1901: 324) and Fisher (1922: 64).

[13] Jevons (1865: 295). For the above criticism concerning Jevons see Padan (1900: 173, 181), and Cooley (1893: 287). Edgeworth even spoke in a footnote of Jevons' "obscure dicta as to the grounds for preferring the geometric mean", cf. Edgeworth (1918: 189). It is clear that, given our present state of index theory, we are now able to say more in favour of Jevons' position.

[14] Es "*hat etwas Bestechendes und wollte auch mich anfangs verführen, allein eine genauere Betrachtung hat mir gezeigt, dass gerade das arithmetische Mittel das richtige ist*" Laspeyres (1864: 96), ("The example appealed to me and at first also almost seduced me as well; only a closer inspection revealed to me that only the arithmetic mean is the correct one").

[15] Laspeyres also modified the example to quantities $q_{1t} = 1$ cwt. cocoa and $q_{2t} = 0$ cwt. cloves (the general equation is of course $4 = 4q_{1t} + q_{2t}$ or simply $200 = \Sigma p_t q_t = p_{1t}q_{1t} + p_{2t}q_{2t}$ where $p_{1t} = 200$ and $p_{2t} = 50$).

As to the reasoning of Jevons, it is notoriously disregarded that + 100 % (in good 1) is canceled by -50 % only when a *subsequent* decline refers to $p_{1t} = 200$, that is to the good of which the price has risen, rather than to the initial price of a second good $p_{20} = 100$. The correct average *over time* (i. e. over a number of adjacent intervals) of a single good i, that is, over p_{i1}/p_{i0}, p_{i2}/p_{i1}, p_{i3}/p_{i2}, … (with a constantly changing base of the relatives), is a geometric mean. However, this has to be kept distinct from an average *over different goods* referring to one interval only (that is, over p_{11}/p_{10}, p_{21}/p_{20}, …, p_{n1}/p_{n0}), in which case it is far from clear that the geometric mean is appropriate.[16]

However, Laspeyres' consideration is also liable to at least two criticisms:

1. What Laspeyres actually defended was not P^C (the index formula he and most of his contemporaries used) but rather the price index of Dutot

$$P^D_{0t} = \frac{\sum p_{it}}{\sum p_{i0}} = \frac{\sum p_{it}/n}{\sum p_{i0}/n} = \frac{\bar{p}_t}{\bar{p}_0} \tag{3}$$

which coincides with Laspeyres' index

$$P^L_{0t} = \frac{\sum p_{it} q_{i0}}{\sum p_{i0} q_{i0}} \tag{4}$$

when all quantities in 0 are equal, as in Laspeyres' example ($q_{10} = q_{20} = 1$ cwt.). That is why he could argue against Jevons in terms of *expenditures*, which always was (and continues to be) a rather popular paradigm[17] of conceiving inflation: a price "level" is rising to the extent that we get less for the same amount of money.

2. Already Pierson 1896 noticed that Laspeyres' example would not have worked so well had he started with unequal prices, for example $p_{10} = 50$ and $p_{20} = 100$, because then $P^D_{0t} = P^J_{0t} = 1$. With $p_{10} = 50$ and $p_{20} = 200$, he even got $P^D_{0t} = 0,8 < P^J_{0t} = 1$, in which case his argument against Jevons had broken down completely. However, Laspeyres apparently had not noticed this because he failed to see that with equal relative price weights P^C boils down to P^D

$$P^C_{0t} = \sum \frac{p_{it}}{p_{i0}} \frac{1}{n} = \sum \frac{p_{it}}{p_{i0}} \frac{p_{i0}}{\sum p_{i0}} = \frac{\sum p_{it}}{\sum p_{i0}} = P^D_{0t}. \tag{5}$$

In his controversy with Jevons, Laspeyres unfortunately only considered *unweighted* arithmetic means. Had he introduced weights α and $1\text{-}\alpha$ in his example he should have seen why his result differs from Jevons'. It is easy to find α for which the weighted arithmetic mean of two price relatives r_1 and r_2, i. e. $P^A = \alpha r_1 + (1-\alpha) r_2$, equals the index P^J of Jevons $(r_1 r_2)^{1/2}$. Assuming $r_1 > r_2$ (and thus $r_2 < \sqrt{r_1 r_2} < r_1$), we get $\alpha = \left(\sqrt{r_1 r_2} - r_2\right) / (r_1 - r_2)$. With $a = r_1 = 2$ and $r_2 = a^{-1} = 1/2$ (Jevons' example), we have $\alpha = 1/3$. The greater a is, the more $P^C = (a^2+1)/2a = (a + 1)^2/2a - 1$ moves away from $P^J = 1$ and the smaller the weight $\alpha = (1+a)^{-1}$ in a weighted arithmetic mean is for which $P^A = P^J = 1$ holds.

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16 Cooley (1893: 287) had already drawn attention to this point.

17 It will become apparent that all indices constructed as ratios of averages (ROA), also known as "generalized unit value" indices, are capable of being interpreted in this way (i. e. in terms of expenditures).

In his controversy with Jevons, Laspeyres referred to the amount of money spent for a certain quantity of commodities instead of solely looking at price relatives.[18] This is clearly another paradigm. It implies that both the type of (selected) goods in question as well as their respective quantities should be the same in the two periods compared (a point Laspeyres repeatedly stressed). This also brings us to the two (not necessarily closely related) problems of defining a price level and assigning weights to prices.

3 Laspeyres and Drobisch: quantities as weights and unit values

Not only Laspeyres but also Drobisch started with the problem arithmetic vs. geometric mean. We will see that Drobisch did not have much to say about this point but instead came up with some new ideas about index numbers. His innovations were twofold:

1. to account for quantities as "weights" assigned to the prices,[19] and
2. to conceive an index as being a ratio of average prices (ROA) rather than an average of price ratios (AOR), which clearly contradicted Jevons and Laspeyres (the latter applied the then prevailing "arithmetic mean" P^C as unweighted and an AOR).

The second point triggered a long-lived controversy (although there are obvious formal[20] relationships between AOR and ROA) that has many ramifications and thus deserves discussion in a separate section (see Sec. 6 below).

3.1 Quantities to account for the relative "importance" of goods

As to the first point, at that time it was by no means clear that weighted means of prices or price relatives are preferable over unweighted means,[21] and secondly if weights were used at all – for example to account for the "relative importance" of goods – it was far from generally accepted that such aspects are best represented by the quantities consumed. Problems of this kind gave rise to a sort of fundamentalism regarding index numbers which is unimaginable for us today. The "importance method" was attacked on the grounds that it would require dubious speculations about satisfaction and perceived well being which have nothing to do with the value of gold[22] (then generally seen as pivotal for prices). Even if quantities were agreed upon (as somehow proportional to "importance"), it was still found worthwhile to discuss:

1. whether explicit quantities q_{i0} or q_{it} should enter the formula or implicit quantities in the form of reciprocal prices would be appropriate, and
2. whether a selection of goods would do, or figures comprising all goods are required.

[18] Not only Laspeyres referred to this notion of an increase of the price level ("inflation" was not yet widely in use) time and again; Oker 1896 and some other authors also expressed it very distinctively.

[19] The British Association (1902: 29) also acknowledges that Drobisch's index had possibly been the first weighted index. The type of weights were called "fluctuating" weights (as they included not only q_0 but in particular also q_t).

[20] Conspicuously, at that time economists (above all in Germany) were habitually not content with purely formal arguments.

[21] The most prominent advocate of unweighted means was clearly F. Y. Edgeworth, but Laspeyres, Jevons and Giffen also repeatedly expressed the conjecture that adding weights might eventually not make a difference. For more details reference can be made in particular to Laspeyres (1883: 797-798) and Walsh (1901: 87-88).

[22] For Pierson (1895: 332) these were "two problems bearing a wholly different character". For him this and the existence of different yet equally reasonable formulas as well as ambiguities with respect to the choice of the base period (we now would say violation of time reversibility) gave reason enough to demand that the system of index numbers "is to be abandoned altogether" (Pierson (1896: 127)).

Ad 1: Implicit *quantities* should be kept distinct from indirect *weights*. The latter is given when one good is represented by a great number of varieties, whereas another good is only represented by one price or no price quotation at all. This was very quickly and fairly generally recognized as a problem of indirect or implicit weighting. Notably Jevons made many experiments with omitting goods or allowing for "importance" by adding more variants (and he concluded that weights would not make a substantial difference).

Implicit quantities (as reciprocal prices) on the other hand are intended to take into consideration that e. g. a pound of silk costs more (viz. p_{s0}) than a pound of bread ($p_{b0} < p_{s0}$). $1/p_{s0}$ and $1/p_{b0}$ are therefore the quantities of silk and bread respectively that are affordable for one currency unit, and relating prices to such "implicit" quantities will provide suitable weights and a common denominator allowing a summation of prices across the board. Many unweighted index formulas, which have now fallen into oblivion, follow this kind of reasoning and account for reciprocal prices of period 0 or t or both periods (see below in Sec. 4 with the index P^Y). This way of looking at "weights" (related to the preciousness of goods rather their physical weight) in the form of "implicit" quantities is an elegant device to give some unweighted price indices a meaningful interpretation in terms of money expenditure and "quantity". It also relates a ROA approach like $P_{0t}^D = \frac{\sum p_{it}}{\sum p_{i0}} = \frac{\bar{p}_t}{\bar{p}_0} = \frac{e_t}{e_0}$ to an AOR approach $\frac{\sum p_{it} \frac{1}{p_{i0}}}{\sum p_{i0} \frac{1}{p_{i0}}} = \frac{\sum p_{it} \frac{1}{p_{i0}}}{n}$ resulting in P_{0t}^C. So P^D (using weights of 1/n) becomes P^C by using weights $1/p_{i0}$, and vice versa; we can easily translate an AOR index like P_{0t}^C into a ROA formula P_{0t}^D (as done in eq. (5)). Implicit quantities q_i^* may also allow an interpretation of $e_t = \sum p_{it} q_i^* = \sum p_{it}(1/p_{i0})$ and $e_0 = \sum p_{i0}(1/p_{i0}) = n$ as a sort of "expenditure".

Nowadays, some authors provide interpretations of unweighted price indices (composed of only prices without any explicit quantities) in terms of substitution behavior which is said to be implied in the formula under consideration. They do so asking which type of weighted index (e. g. Laspeyres, Fisher etc.) will be approximated by such an unweighted index given that prices are sampled with probabilities of selection proportional to quantity shares $q_{it}/\Sigma q_{it}$ or expenditure shares $p_{i0}q_{i0}/\Sigma p_{i0}q_{i0}$ and the like.[23]

Ad 2: A not uncommon view at the time under consideration here was that "quantities" should comprise all sorts of goods (not only actually transacted goods) because money has "power over all goods" (stocks and current production of goods, as well as financial assets),[24] and as it appeared practically unfeasible to provide such weights some people called index numbers "intrinsically impossible"[25] and discarded them as a futile search for the philosopher's stone[26]. Such a wide definition of the "price level" covering all transacted goods (including financial assets) also became popular when some decades later more and more attempts were made to verify the (definitional!) equation of exchange. While such considerations are no longer interesting now, the following aspects of weights are continually relevant.

[23] See for example Balk (2005).

[24] This definitely applies to Lehr (1885: 37) and also to Laspeyres, quite distinctively in (1883: 796): "*Wir haben keine genügende Statistik der durchschnittlichen Konsumtion irgendeines Landes*" [We do not have a sufficient statistic of the average private consumption of any country]). See also British Association (1902).

[25] In German: "*aus inneren Gründen unmöglich*" Held (1871: 321).

[26] Held (1871: 326). Interestingly Held praised Drobisch for the simple reason that he opposed Laspeyres and that this work is good for fostering mistrust in the at that point new method of index numbers.

Once recourse to a *selection* of *explicit* quantities is agreed upon, a decision has to be made on a "single weighting system" (making use of either q_{i0} or q_{it} as suggested by Laspeyres and Paasche) or a "double weighting system"[27] (using both the quantities q_{i0} and the "fluctuating" q_{it} in an index). Drobisch introduced quantities in the concept of a "unit value" of all n goods at time t, defined as follows:

$$\tilde{p}_t = \frac{\sum p_{it} q_{it}}{\sum q_{it}} = \sum p_{it} \frac{q_{it}}{\sum q_{it}} \quad \text{by contrast to} \quad \bar{p}_t = \sum p_{it} \frac{1}{n} \tag{6}$$

and $\tilde{p}_0$ defined correspondingly, which are meant to reflect the price level of a rather comprehensive set of goods. The use of $\tilde{p}$ instead of $\bar{p}$ (as in Dutot's index) has the advantage of avoiding a commensurability problem with respect to prices. Clearly $\bar{p}$ is affected by a move from prices (p_i) quoted in kilograms to prices quoted in pounds ($^1/_2 p_i$), whereas an expenditure $\Sigma p_i q_i$ in the numerator of $\tilde{p}$ is invariant to such changes. The troublemaker is, however, the denominator because Σq_i is in general not defined across all goods. It is difficult, if not impossible, to add over bushels of wheat, tons of iron, yards of cloth and hours of bus rides.[28]

Drobisch felt sure that he had solved this problem properly by requiring that all quantities should be expressed uniformly in hundredweights (cwt., "Centner" in Drobisch's text).[29] This would rule out different results due to isolated changes in only some of the prices but it still does not render the index uniquely determined. As Walsh later pointed out, this is because a change from physical weight in cwt. to another dimension, say bulk [capacity, volume] measured in gallons or cubic meters, again applied to all goods, would yield a different $\tilde{p}$.[30]

Drobisch was not the only author who made use of unit values. Eduard Segnitz (1870) also introduced $\tilde{p}$ as an alternative to the then very popular "midpoint" of prices defined as $(p_{min} + p_{max})/2$ and he was also (possibly unlike Drobisch) aware of the fact that $\tilde{p}_t$ is quite sensitive with regard to the length and position of the time interval t to which its quantities (as a flow variable) refer. It is known from scanner data, now increasingly in use, that it matters a lot whether the q's and therefore $\tilde{p}$ refer to a week or a month and whether the time interval covers some extraordinary events (e. g. sale promotions and the like) or not. [31]

[27] These terms appear to be introduced by Walsh who was, like Drobisch and many others, especially in the Anglo-American index theory, vigorously in favour of a "double system". He wrote (1901: 383) "…the method first discovered by Drobisch of comparing the averages of prices at each period on the mass-quantities of each period, and so employing what we have called *double weighting*". Apart from the double system, which was much to Walsh's liking, however, Walsh had rather a low opinion of Drobisch. Walsh considered the formulas of Drobisch (P^{DR}, eq. 7) and Lehr (P^{LE}, eq. 13) as representatives of double weighting. P^{LE} amounts to taking averages of weights q_{i0} and q_{it} (for each commodity i = 1, …, n), just like Walsh's preferred solution $(q_{i0}q_{it})^{1/2}$ which he called "Scrope's emended method"; Walsh (1901: 540-543).

[28] The nonexistence of such sums over dissimilar quantities is the central shortcoming of unit values.

[29] Note that he did not seek a way to account for the different preciousness of the goods.

[30] Hence unit values are acceptable only for a fairly homogeneous set of goods and thus only for "low level aggregations" and sub-aggregates, that is, as building blocks (taking the part of genuine prices) for greater aggregates. In this sense we have in some countries "unit value indices" (not to be confused with Drobisch's index), especially for the price levels of exports and imports because they are readily available as a by-product of foreign trade statistics.

[31] Segnitz for example maintained that the interval should be neither too short, nor too long. As to experiences with (and the treatment of) scanner data, see Ivancic, Diewert and Fox (2011).

From the definition of $\tilde{p}$ we quite naturally arrive at Drobisch' price index as a ratio of unit values[32]

$$P_{0t}^{DR} = \frac{\tilde{p}_t}{\tilde{p}_0} = \frac{\sum p_{it}q_{it}/\sum q_{it}}{\sum p_{i0}q_{i0}/\sum q_{i0}} = \frac{\sum p_{it}q_{it}}{\sum p_{i0}q_{i0}} \cdot \frac{\sum q_{i0}}{\sum q_{it}} = \frac{V_{0t}}{Q_{0t}^{D}}. \tag{7}$$

Hence as $\tilde{p}$ is the quotient of an expenditure (or a more general value) and a quantity, so P^{DR} is a quotient of the respective indices, V_{0t} and Q_{0t}.

Eq. 7 provides a sort of indirect definition of inflation: less quantity for the same amount of money. P^{DR} may be viewed as an "indirect" (Diewert) or "factor antithetic" (I. Fisher) price index gained by dividing V_{0t} by Dutot's quantity index $Q_{0t}^{D} = \sum q_t/\sum q_0 = \bar{q}_t/\bar{q}_0$.[33]

3.2 Double and single weights

It is beyond the scope of this historically oriented paper to discuss the altogether disappointing axiomatic record of P^{DR}.[34] Since in Drobisch's days great store was generally set by the chain test (transitivity), it is remarkable that P^{DR} is able to comply with this rarely met requirement because $P_{0t}^{DR} = P_{01}^{DR}P_{12}^{DR}\ldots P_{t-1,t}^{DR}$. Most noteworthy is, however, that Laspeyres realized that P^{DR} violates the identity axiom,[35] which requires that a price index should be unity if all prices in t are equal to those of 0. This assumption $p_{it} = p_{i0}$ for all i yields

$$P_{0t}^{DR} = \frac{\sum p_0q_t/\sum q_t}{\sum p_0q_0/\sum q_0} = \frac{Q_{0t}^{L}}{Q_{0t}^{D}},$$

and there is no reason to assume that $Q^D = Q^L$, or (equivalently) that for all goods quantity shares $q_{i0}/\Sigma q_{i0}$ and expenditure shares $p_{i0}q_{i0}/\Sigma p_{i0}q_{i0}$ coincide.

Most importantly, violation of identity implies that P^{DR} does not comply with the idea of "pure price comparison" (a price index should only reflect a price movement), which indeed is the cornerstone of Laspeyres' thinking (see below).

Drobisch was well aware of the fact that his formula specializes to

$$P_{0t}^{L} = \frac{\tilde{p}_t^*}{\tilde{p}_0} = \frac{\sum p_{it}q_{i0}/\sum q_{i0}}{\sum p_{i0}q_{i0}/\sum q_{i0}} = \frac{\sum p_{it}q_{i0}}{\sum p_{i0}q_{i0}} \text{ (Laspeyres price index), and} \tag{7a}$$

$$P_{0t}^{P} = \frac{\tilde{p}_t}{\tilde{p}_0^*} = \frac{\sum p_{it}q_{it}/\sum q_{it}}{\sum p_{i0}q_{it}/\sum q_{it}} = \frac{\sum p_{it}q_{it}}{\sum p_{i0}q_{it}} \text{ (Paasche price index).} \tag{7b}$$

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[32] They are meant as absolute price levels, so P^{DR} is a typical ROA index.

[33] Such an interpretation in terms of the factor reversal test linking a price and a quantity index to the value ratio (see Fisher (1911: 418)) was not familiar to Drobisch, who died in 1896. Also the name Dutot and the concept of a "quantity index" in general was not yet widely known in Drobisch's days. Balk (2008: 7, 73).

[34] For details cf. Balk (2008: 72) and von der Lippe (2007: 18-20).

[35] It is perhaps for this reason that Laspeyres is widely recognized as the "inventor" of this axiom (it is most likely, however, that Laspeyres was not yet aware of the fact that identity is a special case of proportionality). This achievement of Laspeyres and his critique of P^{DR} is also reported in British Association (1902: 30). As pointed out there, the violation of identity (as a disadvantage) may, however, be set against the advantage that P^{DR} can (unlike P^L) reflect substitutions households make in response to changes in relative prices.

What makes the difference between P^L and P^P on the one hand and P^{DR} on the other is that in P^L and P^P reference is made to the same quantities in the numerator and the denominator. This, however, is most important as it avoids the problem with commensurability in P^{DR}. Also, use is made in P^L and P^P of "hybrid" values like $\Sigma p_t q_0$ and $\Sigma p_0 q_t$, which Drobisch considered illegitimate and thus consistently avoided in his formula P^{DR}.

By way of some numerical examples, Laspeyres studied how his index P^L is related to P^{DR}. However, he did not come to conclusions that could be generalized beyond his example. It will be shown here, in an appendix, how these indices are related to one another.

Lehr (1885: 41) also realized that a unit value may indicate a change ($\tilde{p}_t \neq \tilde{p}_0$) although no price in the aggregate has changed, which means that the index P^{DR} violates identity. Lehr therefore rightly maintained that prices are comparable only when the quantities in two periods to be compared either do not differ or at least are proportional[36]

It is interesting to see that Drobisch claimed to be credited with authorship of P^L and P^P, just because both formulas emerge as special cases of P^{DR} (which gives rise to our digression in Sec. 4), although he argued against these formulas, mainly by repudiating somehow hybrid average prices of the type $\tilde{p}_t^*$ and $\tilde{p}_0^*$ as allegedly being logically illegitimate.[37] He did so in his rebuttal of Laspeyres' critique concerning identity,[38] in which he saw an attempt by the latter to place a "death-blow" ("*Todesstoß*" as he put it) to his formula. His reaction was not only peeved but also in no small measure helpless. He argued that Laspeyres might be right "in calculo", but that neither P^C nor P^L is "an authority" for him and that prices in t (and 0 respectively) should not be averaged with quantities other then q_t (or q_0 in the case of p_0). $\tilde{p}_t^*$ is therefore logically illegitimate. He obviously did not see that $\tilde{p}_t^* = \tilde{p}_t$ once the assumption is made. This kind of replying to a "formal" argument by appealing to "logic", "plausibility" and allegedly wrong comparisons is another perennial game in index theory. It was to became very popular, especially in Germany from the 1920s onwards.[39]

Such epistemological issues bring us back to Jevons' choice of the geometric mean. As remarked above, Drobisch did not come to a definite conclusion on this issue, the resolution of which was what was initially called for. In Drobisch (1871b: 154) we find the conjecture that Jevons might have chosen the geometric mean simply because

[36] *Durchschnittswerte (*unit values*) sind "nur unter der Voraussetzung miteinander vergleichbar, daß die Mengen zu verschiedenen Zeiten sich überhaupt nicht oder doch nur im gleichen Verhältnisse änderten*" Lehr (1885: 42) (Unit values "are only comparable under the assumption that quantities at different points in time are unchanged or have only changed in the same proportion"), that is, $q_{it} = \lambda q_{i0}$. This might be understood as an argument in favour of Laspeyres' formula. Lehr's second objection against Drobisch was the commensurability problem with Σq_0 and Σq_t total quantities, which are not even defined across all goods.

[37] He also preferred his formula due its being ostensibly more general, and he erroneously believed that P^L is unrealistic as it requires all quantities to remain constant over time. He apparently seems to have overlooked that the q_0's are kept constant only for analytical purposes, that is, only in a kind of thought experiment.

[38] Apart from this case, to our knowledge Drobisch never entered into discussions about the rightly criticized flaws of his formula.

[39] This refers in particular to the many publications of Paul Flaskämper and his project to develop statistics (and index formulas in particular) solely from reflections on logical conditions of comparability (and mostly without mathematics, or at best mathematics only of the simplest kind). It is beyond the scope of this paper to go into details here, but it is interesting to see why this project was bound to fail.

it yields a lower inflation rate, and also an explicit critique of Jevons which reads as follows: "*Hier vermisst man nun ganz und gar einen* positiven *und* allgemeinen *Grund, aus welchem dem geometrischen Mittel vor dem arithmetischen der Vorzug gebühren soll*" (emphasis by Drobisch. "Here we miss completely a *positive* and *general* reason why we should prefer the geometric mean over the arithmetic."). Furthermore Drobisch correctly pointed out "*…dass aus der Unzulässigkeit des arithmetischen Mittels nicht die Nothwendigkeit des geometrischen folgt, da es ja ausser diesen beiden noch viele andere Arten von Mittelgrößen giebt.*"(p. 154F, "…that from the inadmissibility of the arithmetic mean, it does not follow that the geometric mean must be taken, as there are many more means in addition to these two"). He was also not short of unsolicited epistemological advice addressed to Laspeyres. On the other hand there is much in Drobisch's own writings which may well be criticised. To give an example, Drobisch rejected other formulas as inappropriate or unacceptable for the simple reason that they only coincide under very restrictive and unrealistic conditions with his formula (he studied P^C, P^J and also P^L solely with this intention). The less realistic the assumptions must be in order to approximate his formula, the less meaningful a formula was for Drobisch, as if his formula were the indisputable standard against which everything else should be measured.

4 Digression on priority claims and the re-discovery of formulas (Young's formula)

There are reasons why disputes about priority, such as those that took place between Drobisch and Laspeyres, are sometimes quite difficult to settle. In our view it is not sufficient to realize that P^L (and also P^P) emerges as a special case of P^{DR}, or to indirectly accept both formulas (P^L and P^P) as equally valid or invalid by suggesting a simple arithmetic mean $\frac{1}{2}\left(P^L_{0t} + P^P_{0t}\right)$ of them.[40] In order to claim authorship of a formula it is also desirable if not plainly necessary to demonstrate the comparative advantages of the respective formula over other formulas, and this is precisely what Drobisch clearly failed to do. Furthermore, it was Laspeyres who figured out some shortcomings of Drobisch's formula P^{DR} and thereby advanced arguments to prefer P^L over P^{DR} so that he should rightly be credited for P^L, not Drobisch.

As Kuhn said, "discovery" is a complex process which involves at least two steps: "*that* something is and *what* it is"[41]. He illustrated this fact with the example of the discovery

[40] As done in Drobisch (1871: 425). It noteworthy that in this paper Drobisch was prepared to accept *any* kind of weighted arithmetic mean $\alpha P^L + (1-\alpha)P^P$, not only $\alpha = 1/2$. He was quite indifferent about which α to choose and he suggested this index only in an interrogative sentence: "*Man könnte nun zwar davon das arithmetische Mittel nehmen, welches giebt …, aber muss den der richtige Werth gerade in der Mitte … liegen?*"("One could now possibly take the arithmetic mean, giving …; but why should the true result lie exactly in the middle between the two?" Drobisch (1871c: 425)). Interestingly Drobisch not only saw no reason to prefer one formula over the other, he also made use of "crossing" of formulas which later became very fashionable (Irving Fisher in particular made extensive use of it in his index theory). In the Anglo-American literature the above mentioned index Drobisch suggested (for $\alpha = 1/2$) is also known as the Sidgwick–Bowley index (see e. g. Diewert 1993 for more details). Also v. Bortkiewicz (1932: 24) remarked that the index $1/2(P^L + P^P)$ should not be credited to ("the philosopher") Drobisch – just because of his quoted indifference regarding the choice of $1/2$ for α – and he also said that the formula was proposed by Henry Sidgwick (an English utilitarian philosopher 1838 – 1900).

[41] Kuhn (1996: 55).

of oxygen. There were at least three claimants of authorship: Scheele, whose experiments led him to infer its existence but who was too hesitant to publish his finding in time; then a bit later Priestley, who was the first person able to isolate this gas but was unable to understand its real nature; and finally Lavoisier, who after having received hints from Priestley was the first who analyzed and understood (almost) correctly what kind of gas it was.

The lesson to be learned from this example is that claims of priority are more often than not questionable. In addition, a potential forerunner is sometimes less precise and delivers only a more-or-less vague verbal description rather than a formula. For example, Jastram 1951 observed that Willard Phillips might be called a predecessor of Paasche because he wrote in his Manual of Political Economy in 1828 that an absolute (constant) measurement rod of value should not be strived towards and could not be established in the form of labour (which was generally accepted by his contemporaries). Instead, the standard of value should be different for different times, and depend on prices of goods. Phillips also suggested that "quantities of the different articles assumed ought to be in proportion of the consumption or the amount possessed in the country or district for which the measure is framed."[42] Phillips also noticed that substitutions will make q_0–weights inappropriate: "Without changing the amounts of articles to correspond to the differences of consumption, the table would not be a fair representation." This may be understood in such a way that Phillips "table" should include quantities q_{it} rather than q_{i0}. However a "table" (not even worked out empirically, and not reduced to a ratio of expenditures) is not yet a formula which in turn is more than just a (suggested) list of q_{it} quantities.

We can also easily quote various remarks which can be interpreted "with hindsight" as a very early allusion to the "economic theory of index numbers". Such words can even be found in the writings of authors who plainly rejected index numbers altogether.[43] Yet such more-or-less vague and only occasional remarks are far from anticipating the mathematically developed economic approach as presented for the first time only as recently as 1924 by Konüs.[44] To my knowledge it is due to von Bortkiewicz that his work became known beyond Russia.[45]

It not only happens that somebody has dubious claims of priority and an *alleged* authorship, it may also easily happen that authorship is erroneously claimed because something new (subjectively) is found without knowing of a *real* predecessor and therefore independently of him. Such "re-discovery" is not unusual, even today. The following provides an example of "rediscovery" in index theory. Allyn Young (not to be confused with Arthur Young 1812) proposed the following seemingly weird and unmotivated formula of Young (1923: 357) which reads as follows:

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[42] Jastram (1951: 125).

[43] This applies for example in Germany to Held (1871: 331) for whom with inflation the question was whether "..*noch die alten Bedürfnisse im alten Umfang oder nur in geringerem Umfang befriedigt werden können*" (...we can continue to satisfy our needs to the old extent or only to a limited degree) and interestingly he concluded (just like Paasche 1878) that much of what seemed to be inflation was only a self-deceit due to the growing needs of consumers.

[44] Held, who was an engrained skeptic as regards mathematics in economics and particularly index numbers (as were many others in Germany at that time) was of course light years away from Konüs.

[45] See also von Bortkiewictz (1932: 18), where he quotes the original Russian text of Konüs.

$$P_{0t}^{Y} = \frac{\sum p_t \sqrt{\frac{1}{p_0 p_t}}}{\sum p_0 \sqrt{\frac{1}{p_0 p_t}}} = \frac{\sum \sqrt{\frac{p_t}{p_0}}}{\sum \sqrt{\frac{p_0}{p_t}}}, \tag{8}$$

and which Irving Fisher (1927: 530 f.) later called "an ingenious anomaly, scarcely classifiable" (in the scheme of Fisher's book) and "a scientific curiosity". Not surprisingly, it soon fell totally into oblivion, possibly also because the derivation of P^Y was not well understood, although it can easily be explained. Using implicit quantities (by way of inverse prices), Young found that "base year weighting" in $P_{0t}^{C} = \sum p_t \frac{1}{p_0} \Big/ \sum p_0 \frac{1}{p_0}$ "overweights rising prices", by contrast to $P_{0t}^{H} = \sum p_t \frac{1}{p_t} \Big/ \sum p_0 \frac{1}{p_t}$, the harmonic mean, which tends to underweight them. Thus he was quite naturally led to the geometric mean $\sqrt{\frac{1}{p_0 p_t}}$ as a compromise.

The formula was then rediscovered by Bert Balk, who called it the Balk-Walsh index,[46] because with *explicit* quantities we obtain $P_{0t}^{W} = \frac{\sum p_t \sqrt{q_0 q_t}}{\sum p_0 \sqrt{q_0 q_t}}$, i.e. Walsh's formula, as the weighted counterpart. The geometric mean of P^C and P^H, called the CSWD-index[47] is also known to approximate P^J fairly well.

Another rediscovery of P^Y took place when Jens Mehrhoff – in a short note he contributed to von der Lippe (2007: 45 f.) – looked for a linear index able to approximate P^{CSWD} and thereby also P^J. He called it "hybrid index", and later the BMW-index (Balk Mehrhoff Walsh), not knowing that it coincides with P^Y.

Young also saw that his index meets the time reversal test but not the circular test, which means that P^Y is not transitive

$$P_{02}^{Y} = \frac{\sum \sqrt{\frac{p_2}{p_0}}}{\sum \sqrt{\frac{p_0}{p_2}}} \neq \frac{\sum \sqrt{\frac{p_1}{p_0}}}{\sum \sqrt{\frac{p_0}{p_1}}} \frac{\sum \sqrt{\frac{p_2}{p_1}}}{\sum \sqrt{\frac{p_1}{p_2}}}, \tag{8a}$$

and finally he also noticed (interestingly in view of Mehrhoff's intentions that led him to P^Y) as regards P^Y that, "In general it will agree very closely with the geometric average" (357) i.e. with P^J.

5 Laspeyres and Paasche: single quantity weights (q_0 or q_t?) and "pure" comparison

It has often been stated (approvingly for example by Walsh) that Drobisch's formula (as well as Lehr's below) may be viewed as a double weighting formula while P^L and P^P represent formulas with single weights only. Today, so called "symmetric" index formulas in the sense of price index functions $P(\mathbf{p}_0, \mathbf{q}_0, \mathbf{p}_t, \mathbf{q}_t)$ that treat price and quantity vectors of both the period 0 and the period t in a symmetric manner (such as Fisher's or Törnqvist's index) are often viewed (e.g. by Diewert) as being superior to indices like P^L and P^P that only make use of either $\mathbf{q}_0$ or $\mathbf{q}_t$ respectively. Symmetric indices particularly stand out in relief against other indices because all "superlative" indices

[46] Balk (2005 and 2008: 187-191).
[47] Proposed by Carruthers et al. (1980) and Dalen (1992); see Balk (2008: 184).

(i. e. indices able to approximate the "true cost of living index", or "constant utility index" in the sense of the economic index theory) are symmetric, such as for example the indices of Fisher, Walsh and Törnqvist.[48] The definition of "symmetry" applies to Drobisch's index P^{DR} as well, but P^{DR} is far from being superlative.

Of course the notion of superlative indices was unknown in the 19th century and it was definitely not the intention of Prof. Laspeyres or Prof. Paasche to provide an upper or lower bound to the cost of living index, and so it is better to restrict ourselves in the discussion of pros and cons of formulas to those ideas that were already known in the late 19th century.

Even before the economic index theory became influential, the proponents of double weights seemed to have prevailed over the "single weighters" and it fits to their view that they consider the P^L and P^P indices to be equally well reasoned.[49]

This being the situation, "single weighters" have always had a hard job. Many theories have been advanced as to why Laspeyres insisted upon q_0-weights and Paasche on q_t-weights. Interestingly, both were conspicuously taciturn as regards this issue. In his controversy with Drobisch, Laspeyres confined himself to exploiting the comfortable position that he could quote Drobisch (although both men obviously disliked one another considerably) for this purpose.[50] He apparently thought that this would be disarm Drobisch and save him the trouble of substantiating his position.

It is often stated that Laspeyres only took quantities q_0 for practical reasons,[51] and that he would have taken q_t (or q_t in addition to q_0) if only he had better access to timely data on such quantities. Lack of suitable data were admittedly the reason for initially only using the unweighted P^C-index, however, it is far from clear that he would have preferred a constant and timely update of weights, or even a double weighting system, if only he had had access to appropriate data.

Laspeyres (1883: 796) is one of the rare occasions where he discussed – explicitly referring to Conrad and Paasche – the problem of whether q_0, q_t or some average of both should be taken. He concluded "*Doch sind dies praktisch noch unzweckmäßige Fragen*" ("However, these questions are still unsuitable from a practical point of view"). He was referring to the state of statistics on *all* quantities produced and consumed. As already mentioned, he was obviously misled by the then common belief that such quantities ought to refer to the whole economy rather than a sample of consumers. And as he saw that he was unlikely to get such statistics,[52] he decided to pay more attention to

48 More about the notions "symmetric" and "superlative" index functions cf. Diewert (1976).

49 "Nothing can be offered in proof of the superiority of the one over the other" (Walsh as discussant in Fisher 1921: 538), a statement which may serve as backing of the widely held view that some kind of average of the two indices (like Fisher's "ideal" index $(P^L P^P)^{1/2}$) should be taken. There was a discussion in Germany in the late 19th century about whether or not to average index functions like P^L and P^P or to average weights (q_0 and q_t). We will come back to this at the end of this section. At the moment our focus is on Laspeyres' position (as opposed to Drobisch).

50 He quoted Drobisch (1871b: 145): "*Wir nehmen dabei, zur Vereinfachung an, dass seine Lebensbedürfnisse in qualitativer Hinsicht sich gleich geblieben sind, und auch quantitativ sich weder vermehrt noch vermindert haben.*" (To make things easier we assume that needs did not rise nor fall, neither quantitatively nor qualitatively.)

51 Roberts (2000: 10).

52 The exceptions he saw were import statistics and consumption patterns of working class households, possibly based on a sample, because the variability of such patterns will tend to be smaller than for better-off families.

other points, especially a justification of unweighted means like P^C which he continued to prefer over his own formula for many purposes (e. g. measuring the purchasing power of money).

In other contexts, however, we can see clear indications that he was not indifferent concerning q_0 or q_t. There are good reasons to assume that he had deliberately kept weights constant even for a relatively long interval in time and even if availability of data could have enabled him to do otherwise. Constant quantities were essential to him as a device to imitate and simulate an experiment as the only way to prove causality. He was always immensely interested in both causal inference[53] and the prerequisites needed for making valid comparisons. Ideally, statistical figures should reflect a hypothetical and "pure" process as a surrogate of an experiment. Obviously for him constant quantities were an artificial "*ceteris paribus*" that permit isolating the factor "price" from other correlated variables and influences such as demand, income etc. Constancy is not meant as a counterfactual description of a real process but rather as a kind of model, intended to achieve in the social sciences something analogous to an experiment in the natural sciences.[54] The underlying idea is particularly clearly spelled out in his paper on "*Kathedersocialisten*"(Laspeyres 1875), where he wrote:

> *Um "den Charakter der Bewegung kennen zu lernen muß man nicht vorwärts, sondern lieber rückwärts schauen und diejenigen Objecte aussuchen, welche ausnahmsweise eine lange Zeit in vergleichbarer Qualität producirt wurden"* (p. 18).[55] He demanded that these objects "*in die* Vergangenheit *recht weit zurückverfolgt werden können*" (it should be possible to trace them back fairly far *into the past*; emphasis original). And finally he said as a kind of credo:
>
> "*Die statistische Untersuchungsmethode kann einen Schritt weitergehen, sie* nimmt nicht an, *daß die anderen Umstände alle gleich seien, sondern sie* macht *alle anderen Umstände gleich, mit Ausnahme des einen, dessen Wirkung sie untersuchen will, den einen Umstand aber, dessen Wirkung sie untersuchen will, macht sie möglichst verschieden ...* " (p. 32).[56]

The reason for using quantities of the base period in a number of subsequent periods, which is the characteristic feature of P^L, can be seen in the sequence

$$P_{01}^{L} = \frac{\sum p_1 q_0}{\sum p_0 q_0}, P_{02}^{L} = \frac{\sum p_2 q_0}{\sum p_0 q_0}, P_{03}^{L} = \frac{\sum p_3 q_0}{\sum p_0 q_0}, \dots \tag{9}$$

[53] He introduced his "mammoth number-crunching" work (Roberts), which is Laspeyres 1901, and which kept him busy for many years with the question "*Kann man statistisch ein post hoc als ein propter hoc nachweisen?*" (Can you prove statistically a "because" with observations of the "after" type?).

[54] This idea is rejected with much vigour in Winkler (2009: 101-110), who in a way represents the very opposite of Laspeyres and recommends going back to Dutot's index.

[55] This means: "In order to understand the character of a movement you should not look ahead but rather backwards and choose such objects that coincidentally are produced for a long time in comparable quality." In the light of some other statements it seems to be fair to say that Laspeyres also would have emphasized "long".

[56] "The statistical experiment method can go one step further (than theory), it does not *assume* that everything else remains constant, it rather *makes* all other circumstances constant with the exception of the one whose effects it wishes to test, which is made as different as possible."

in which subsequent indices differ only with respect to prices. By contrast for P^P

$$P^P_{01} = \frac{\sum p_1 q_1}{\sum p_0 q_1}, P^P_{02} = \frac{\sum p_2 q_2}{\sum p_0 q_2}, P^P_{03} = \frac{\sum p_3 q_3}{\sum p_0 q_3}, \ldots \tag{9a}$$

we get continually changing weights q_1, q_2, q_3, ... Prices in this sequence (as opposed to P^L) are therefore not comparable among themselves but only to p_0.

There were (and still are) not many people who clearly distinguish between a "year-on-year" (or bilateral) comparison only, and a "comparison-in-series", as Young (1923: 364) put it. In the former situation P^P may be as good as P^L (i.e. what applies to 0 in P^L simply applies to t in P^P, and t is one period just like 0), whereas in the case of a series t denotes not one period (like 0) but a number of periods (t = 1, 2,...) and there P^L may well be preferred over P^P from the point of view of "consistent series" (Young), or "pure price comparison",[57] a concept which is more difficult to define in exact terms than might appear at first glance.[58] While q_0 is kept constant (for some periods), q_t is "fluctuating", constantly changing with the passage of time.

Of course there were soon critics of the P^L formula in abundance, and they quickly got into the habit of deriding P^L predominantly because of its constant weights. This has continued to be the standard argument ever since, and it goes as follows: Keeping the selection of goods and their weights constant is difficult in a dynamic economy and results in the index sooner or later hopelessly losing touch with reality.

As to Paasche, the situation is quite different. It is difficult to find pronounced statements as to why he preferred weights q_t over the weights q_0 , and why he did not chose both q_0 and q_t.[59] We can find statements concerning the first point, but few (if any) concerning the second.

There are remarks on the part of Paasche that were intended to justify the preference for a single weight system (which also would apply to q_0 instead of q_t), intended in the first place to avoid ambiguity of the index (reflecting possibly both price *and* quantity movement). Similar arguments can also be found in the writings of Johannes Conrad, a long time editor of this journal (1878 – 1915) and a promoter of Paasche and many other authors in index theory.[60]

[57] We could not find suitable quotations published in Laspeyres' time but only some fifty years later. In addition to Young, Persons also had a similarly clear position (and explicitly advocated P^L) as a discussant in Fisher 1921. He said there: "An index number is not computed merely to compare the index number of one given year with that for the base year, but to compare the indices for a series of years with each other ..." (p. 545). He definitely argued in favour of pure price comparison, because for him double, and therefore *variable* weighting, "has the defect that we do not know whether changes in the indices result from changes in the prices or production" (p. 545). Conspicuously Fisher made clear in his rejoinder that the "time reversal test" (on which he and many others laid a disproportionate stress) rules out all indices "which do not have symmetrical or 'double' weighting" (p. 549), and that Person's position is inconsistent and amounts to "demanding the impossible".

[58] Cf. von der Lippe (2005) (the whole paper is on the problem of defining "pure comparison").

[59] Lehr (1885: 44) argued that Paasche wanted the same quantities in numerator and denominator in order to avoid problems (of P^{DR}) with commensurability. However, it is not double weighting that ensues commensurability problems but rather the summation of quantities (be they q_{i0} or q_{it}) *over n > 1 goods* (i = 1, ... , n). Lehr's own index (see below) makes use of q_{i0} *and* q_{i1} but the summation takes place over periods (0 and 1) for each good i separately (just like the averages $(q_{i0}q_{it})^{1/2}$ in Walsh's index) so that no commensurability problems can arise.

[60] Among them for example the American Samuel McCune Lindsay, who (like Paasche) received his Ph. D from Conrad in Halle and whose book on prices (in German) was extensively commented by Edgeworth (1894).

However, Paasche also expressly stated that he could well imagine that a double system would make sense. He mentioned possible studies of how consumption changed and households escape inflation by substituting goods, but in a nebulous and not at all satisfactorily substantiated remark he concluded that taking both periods into accounting would not be advisable:

> "*Aber für die einfache Constatierung und Berechnung der Preissteigerung würde das wenig helfen, weit mehr verwirren, denn das sind allerdings wesentliche Momente für die Bestimmung der Preise, aber für die Aufstellung des Verhältnisses der einmal gegebenen Werthe nicht weiter von Einfluß.*" Paasche (1874: 173).[61]

In summary his position was: quantities of the past q_{t-1}, q_{t-2}, … may be interesting as factors determining the present situation, but they should not be taken into account when the task is to establish a price index comparing 0 to t. This of course caused him some trouble with Lehr.

It is true that he gave detailed comments on the plausibility of his empirical index calculations as regards specific commodities, but he gave only sparse comments, if any, on why he preferred q_t over q_0. The motivation was possibly (as it always is in the standard critique of P^L) only that more recent and constantly updated quantities are considered better.[62]

A final remark relating to P^L vs. P^P and single vs. double weights may be in order:[63] Drobisch felt irritated by the fact that both formulas (P^L as well as P^P) are equally possible (or perhaps even equally plausible) and he requested an unequivocal solution which he believed to have found in his formula P^{DR}.[64] The problem of P^{DR}, however, is that it fails identity, as Laspeyres rightly noticed. Avoiding this seemed to require a single weighting system,[65] which on the other hand requires making a choice between q_0 and q_t. Lexis proposed (as an "improvement" of Drobisch's method) to make use of $1/2(q_0 + q_t)$, which results in a formula now known as the Marshall-Edgeworth index.[66] Lexis also viewed

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[61] The quotation reads as follows: "However, this would not be helpful for the simple identification and quantification of a price increase, but rather cause confusion, because these aspects may be relevant as determinants of the present prices, but have little influence on relating the given prices (to the past by way of an index, he meant)." Not surprisingly Lehr (1885: 44) quoted (most disapprovingly) precisely this very sentence, especially as regards the (alleged) "confusion". Lehr's message – we will see – was essentially that what was called for was not a binary comparison (0 to t), but rather a time series in which all intermediate periods are to be taken into account.

[62] Later van der Borght (1882), who continued Paasche's and Conrad's regular compilations of price statistics in this journal, argued in a similar vein in favour of P^P: This index is more convenient when it is difficult to find prices in t which match with those in 0, as with P^P there is no need to look back in time. This kind of reasoning was, and still is, notably popular among all those who advocate chain indices. Richard van der Borght later became president of the German Imperial Office of Statistics (1904 – 1912).

[63] The following chain of reasoning is nicely developed in Lexis (1886: 117-121).

[64] As mentioned above (Footnote 40), it is therefore not quite correct to credit Drobisch for the formula $1/2(P^L + P^P)$ although he took it into consideration.

[65] According to Lexis (1886: 118), Paasche advocated the single weight system even more than Laspeyres did.

[66] The German text introducing the averaged weights $1/2(q_0 + q_t)$, reads as follows: "*daß man für jede Ware den Durchschnitt aus der verkauften Menge des Anfangs- und des Endjahres in Rechnung brächte*" Lexis (1886: 119). I only discovered this paper of Lexis thanks to a quotation of von Bortkiewicz (1932: 24) who also maintained that the formula was proposed in 1886 by Lexis and thereafter also recommended by Marshall and Edgeworth.

the formula of Lehr (eq. 13a) as an attempt to resolve the dilemma of choosing among two weights. For Lehr a (price) weighted average between quantities of adjacent periods appeared to be the solution[67] which quite naturally made Lehr advocate chain indices. So there are various possibilities that may lead us to plea for chain indices, e.g. a choice among weights (Lehr), ambiguities with non-time-reversible formulas which makes independence of the base period attractive (an argument used for example by Pierson (1896: 128), or Flux (1907: 618)), and perhaps above all practical problems with new and disappearing goods and the fixed weights q_0 becoming progressively obsolete with the passage of time (a point made by many authors, among them Lehr). We will discuss Lehr's approach in more detail later (Sec. 7).

6 Average of price relatives (AOR) or ratio of average prices (ROA)

We have already introduced the distinction between the ROA approach, of which Drobisch's P^{DR} is an example, and the (before Drobisch prevailing) AOR approach (as in the form of P^C and P^J). The dichotomy triggered a host of ultimately useless controversies as early as in the 19th century and has continued to do so ever since. The problem for proponents of the ROA approach[68] is that they are tempted to view price indices (comprising $n > 1$ goods) by analogy to simple price relatives (each in turn referring to only one good) and thus to demand that indices fulfill all those axioms that relatives necessarily fulfill. This applies in particular to transitivity, but also to Fisher's reversal tests, which were obviously patterned against the model of simple relatives.

Proponents of the AOR view, e.g. Jevons, habitually hesitate to aggregate over prices referring to such different quantity units as hours, cwt, gallons etc., but they have no problems with the same figures when transformed into relatives (as they then become dimensionless pure numbers). Getting rid of such problems with dimensions was seen as a main advantage of AOR.[69]

[67] Lexis (1886: 118) criticized the choice of only two adjacent periods, that is q_0/q_1, q_1/q_2, … (he asked: why not average over more than just two periods?) and for von Bortkiewicz (1932: 31) the problem with Lehr's formula is that it violates proportionality (while identity is satisfied) and Lehr's erroneous interpretation of his formula in terms of "utility". He also did not endorse the chain approach of Lehr. In Bortkiewicz's view Walsh and Edgeworth were also too cautious and indulgent in criticizing Lehr. See also Walsh (1901: 386, 547), Edgeworth (1894: 160) and Edgeworth (1901: 404) for their views on Lehr.

[68] In Germany for example the previously-mentioned Paul Flaskämper was a crusader for the cause of ROA indices, which he considered the only "logically" tenable indices. For him the only difference between an index and a relative is that the former has an *average* of prices rather than a single price in its numerator and denominator and thus (the conclusion is far from convincing as it denies all aggregation problems) should share all properties with relatives. He even went so far as to deny the relevance and validity of so many simple equations which show that the two approaches are often quite closely related. This is perhaps again a consequence of the then prevalent propensity to philosophize in German statistics (a major exception was L. von Bortkiewicz). We cannot go into more details here concerning the problem with Paul Flaskämper (1928) and the decline of German index theory in that period. It will be the subject of another paper.

[69] With this motivation e.g. Irving Fisher was perhaps the most prominent advocate of the AOR approach. "My book is devoted entirely to averages of ratios" Fisher (1923: 743). Some proponents of AOR were quick in ridiculing their ROA opponents: "Actually a price index number is not properly thought of as a ratio of average prices. An average of the number of horses and the number of apples has little, if any, meaning. Neither has an average of the price of horses and the price of apples. An index number should be an average of ratios, not a ratio of averages." Cowden and Pfouts (1952: 92).

It may seem strange that the alternative of AOR vs. ROA stirred up so many controversies, since many index functions can be written in both ways. We already demonstrated this with the translation of P^D (ROA) into P^C (AOR) and vice versa simply by introducing weights. It seems to be very well known that the two formulas exist for P^L

$$P^L_{0t} = \frac{\sum p_t q_0}{\sum p_0 q_0} = \sum \frac{p_t}{p_0} \frac{p_0 q_0}{\sum p_0 q_0}.^{70} \tag{10}$$

As to Young's index P^Y in the digression: Mehrhoff also remarked (like v. Bortkiewicz 1927 beforehand) that P^Y not only has a ROA interpretation (indicated in (8) with weighted means of prices), but also an AOR interpretation, namely $P^Y_{0t} = \sum \frac{p_t}{p_0}\sqrt{\frac{p_0}{p_t}} \Big/ \sum \sqrt{\frac{p_0}{p_t}}$, and thus is indeed classifiable in Fisher's scheme, which Fisher apparently failed to see.[71] It may be noticed in passing that Young well appreciated this double interpretation of his index P^Y.[72] This is what L. v. Bortkiewicz (1927: 747) elevated to the rank of a quality indicator in the form of his "*Zwieförmigkeitskriterium*"[73]. It may be viewed as an axiom which (by way of exception, given the nature of the other axioms) directly focuses on "meaningfulness" and "understandability" of a formula.

There are of course indices which allow only one interpretation. P^{DR} is for example not a weighted mean of relatives for sub-aggregates. Assume K sub-aggregates and "partial" unit values $\tilde{p}_{kt}$ and $\tilde{p}_{k0}$ (k = 1, …, K). The ratio of unit values $\tilde{p}_{kt}/\tilde{p}_{k0}$ is not a mean of price relatives because $\frac{\tilde{p}_{k1}}{\tilde{p}_{k0}} = \frac{Q_{k0}}{Q_{kt}} \sum_j \frac{p_{kjt}}{p_{kj0}} \left(\frac{p_{kj0} q_{kjt}}{\sum_j p_{kj0} q_{kj0}} \right)$ and the aggregated P^{DR} is not simply a weighted mean of the $\tilde{p}_{kt}/\tilde{p}_{k0}$ ratios because $P^{DR}_{0t} = \sum_k \frac{\tilde{p}_{kt}}{\tilde{p}_{k0}} \left(\frac{\tilde{p}_{k0} \sigma_{kt}}{\sum_k \tilde{p}_{k0} \sigma_{k0}} \right)$, where the σ are quantity shares $\sigma_{kt} = \sum_j q_{jkt} \Big/ \sum_k \sum_j q_{jkt}$ and σ_{k0} defined correspondingly. Hence there is no AOR interpretation of P^{DR},[74] whereas both of the indices P^L and P^P, which Drobisch regarded as special (and inferior) cases of his formula, can be interpreted in both ways, i.e. ROA and AOR. The problem with P^L is of course the fictitious character of an average price $\tilde{p}^*_t = \sum p_t \frac{q_0}{\sum q_0}$ and an expenditure $\sum p_t q_0$ requiring that recourse has to be made in period t to quantities q_0 in the past. This brings us to another perennially contentious issue: chain indices.

[70] That indices such as P^L (and also P^P) can be written in both ways had already been seen by Walsh (1901: 428, 539) and Fisher (1911: 365). Many students seem to be unaware of the fact that two such formulas exist for other indices than P^L (for P^P for example) as well, and they tend to mystify the fact that prices are multiplied by (absolute) quantities, while price *relatives* are multiplied by expenditure shares. This was also a problem that appeared puzzling and vexing to Flaskämper and other German statisticians of his time.

[71] Fisher's weights were, however, expenditure shares, not square roots of reciprocal prices relatives.

[72] "In a way Professor Fisher is right in holding that all true index numbers are averages of ratios.' But I should prefer to say that all true index numbers are at once averages of ratios and ratios of aggregates." (Young 1923: 359).

[73] For him this "two-way" (or twofold) interpretation had the rank of an axiom or test, just like time reversibility or proportionality. Note that there are examples of indices which allow both interpretations (P^L and P^P), only one (as P^{DR}), or none of them (unless in a quite farfetched manner), such as e.g. Fisher's highly esteemed "ideal index" P^F. Hence index theorists will most definitely disagree on Bortkiewicz's "*Zwieförmigkeit*" (existence of two forms) lest P^F will be downgraded for its poor performance in this respect.

[74] This also applies to Lehr's index P^{LE} in the next section.

7 Chain indices, Lehr and the ideal of "pure price comparison"

The idea of chain indices arose not only from the conviction that weights need to be continuously updated, but perhaps (as mentioned above) to an even greater extent from the undue embarrassment with contradictory empirical results of time series of index numbers when they referred to different base periods.[75] Chain indices were welcomed as a device to solve the problem of choosing a base (by finding means to be independent of the base), to avoid ambiguities in this respect and to update weights (which P^L fails to do).[76]

A chain index is defined as a product of indices ("links"), each of which refers to two adjacent periods (as a short sub-interval). To arrive at an index for the total interval, the links are multiplied to form a chain. When the index is transitive, such as for example P^{DR}, it is clear that the direct index coincides with the chain $P^{DR}_{0t} = \bar{P}^{DR}_{0t} = P^{DR}_{01} P^{DR}_{12} \dots P^{DR}_{t-1,t}$. The same applies to Jevons' unweighted index P^J.

However, chaining of an index also takes place when the respective index is not transitive, which applies for example to P^C or P^L, where therefore as a rule $P^L_{0t} \neq P^L_{01} P^L_{12} \dots P^L_{t-1,t}$.[77] Usually much of what is argued in favour of chain indices grows out of a critique of so called "fixed weight" or "fixed base" indices like P^L. This seems to apply to Lehr too.

It should be noticed that the terms "fixed weight" or "fixed base" are incorrect and should be abandoned. This can easily be seen in the case of a sequence of a *"fixed base" Paasche* price index (see eq. 9a), where the q-weights are not fixed but instead constantly vary in much the same way as in the case of factors creating a *chain Paasche* price index given by

$$\bar{P}^P_{0t} = \frac{\sum p_1 q_1}{\sum p_0 q_1} \cdot \frac{\sum p_2 q_2}{\sum p_1 q_2} \cdot \dots \cdot \frac{\sum p_t q_t}{\sum p_{t-1} q_t} \neq P^P_{0t} = \frac{\sum p_t q_t}{\sum p_0 q_t}. \tag{9b}$$

The correct characterization should be chain index by contrast to direct index, because the alternatives are either to compare 0 to t indirectly via 0-1, 1-2, … t-1,t or to compare 0 to t directly.

The property of a chain index that is particularly often found desirable is to always have "realistic" (up-to-date) weights by constantly switching to more recent quantity-weights. However, this is clearly in conflict with making "pure comparisons" by keeping

[75] As mentioned above, such puzzles led many economists (for example Pierson) to a general rejection and ostracism of all sorts of index numbers. In the time period under consideration it was not uncommon to take averages over *a number of* years (e.g. five or even ten) as the base period ("standard") in order to mitigate potential "resounding" effects of an inappropriate single base year.

[76] Interestingly, until recently most of the arguments in favour of chain indices were advanced very early on and have remained by and large the same until today. As many others, Flux (1907) for example was obsessed with quantity weights being as up-to-date as possible, because fixed weights get "thoroughly out of touch with the facts" (p. 619). Furthermore, chain indices or the "method of year-to-year steps" as he called it, has the advantage of being "not dependent on the location of the starting point" and to facilitate "introducing new articles or dropping old ones" (p. 625). Much the same can be read in Fisher's writings. Such arguments have continued to dominate all debates of chain indices vs. Laspeyres' formula ever since. For more details see von der Lippe (2001).

[77] It makes sense to call the left hand side of this inequality the "direct index" because it compares t directly to 0 without taking into account the intermediate periods. However, it is unfortunately more common to speak of a "fixed base" index as opposed to a chain index.

q_0 constant. Chain indices can be viewed as an option in favour of a constant update at the expense of making "pure comparisons". Reconciling both advantages appears to be insoluble.[78]

Diewert mentioned Marshall and Lehr in the context of chain indices, and von Bortkiewicz wrote that such indices were first suggested by Lehr and then by Alfred Marshall.[79] It may therefore be pertinent to briefly present the index theory of Julius Lehr.

As mentioned above, unlike Laspeyres, Paasche and Drobisch, Lehr did not contribute papers on index numbers to this journal. He developed his somewhat peculiar formula (denoted in the following by P^{LE}) in a small pamphlet (Lehr 1885). In this book, however, P^{LE} covered only a couple of pages, and it can be seen that he multiplied the respective links P_{01}^{LE}, P_{12}^{LE}... as if this were a matter of course, but he did not say much about the properties of a chain index.

Before going into details of how Lehr justified chaining, we should introduce his formula which is, like P^{DR}, a typical ROA approach. Central to this index is the fictitious quantity $g_{i,01}$, called "*Genußeinheit*" (or "pleasure unit" in the translation of Edgeworth[80]) by Lehr. In the tradition of implicit quantities it is conceived as a reciprocal price level $1/\breve{p}_{i,01}$. However, $\breve{p}_{i,01}$ depends on explicit (effective) quantities q_i and combines the prices of two periods. In

$$\breve{p}_{i,01} = \frac{1}{g_{i,01}} = \frac{p_{i0}q_{i0} + p_{i1}q_{i1}}{q_{i0} + q_{i1}} = p_{i0}\frac{q_{i0}}{q_{i0} + q_{i1}} + p_{i1}\frac{q_{i1}}{q_{i0} + q_{i1}} \tag{11}$$

we may see a sort of mid-interval price of good i because averaging takes place over two adjacent periods in time, and not over two goods. This leads to Lehr's definition of an absolute price level[81]

$$\breve{P}_1 = \frac{\sum_{i=1}^{n} p_{i1}q_{i1}}{\sum_{i=1}^{n} q_{i1}\breve{p}_{i,01}} = \sum p_{i1}\frac{q_{i1}}{\sum q_{it}\breve{p}_{i,01}} \text{ in period 1, and } \breve{P}_0 = \frac{\sum p_{i0}q_{i0}}{q_{i0}\breve{p}_{i,01}} \tag{12}$$

correspondingly in period 0, so that his index as a ratio of price levels is given by

$$P_{01}^{LE} = \frac{\breve{P}_1}{\breve{P}_0} = \frac{\sum p_{i1}q_{i1}}{\sum p_{i0}q_{i0}} \cdot \frac{\sum q_{i0}\breve{p}_{i,01}}{\sum q_{i1}\breve{p}_{i,01}} = \frac{\sum p_{i1}q_{i1}}{\sum p_{i0}q_{i0}} \cdot \frac{S_0}{S_1} = \frac{V_{01}}{Q_{01}^{LE}}, \tag{13}$$

[78] This dilemma was made particularly clear in a paper of Sir George Knibbs. On the one hand we can find a plea for an "unequivocal" index (a "price index of an indefinite or variable basis cannot possibly yield an unequivocal result" and "the whole purpose of a price-index is to reflect the effect of change of price solely..." Knibbs (1924: 46)), and on the other hand a plea for representativeness (what he called "reality"). For Knibbs the conflict between pure price comparison (what he called "definiteness") and reality is "the crux of the whole matter" (p. 60), a problem of the squaring the circle type. He called a price index which also reflects quantity movement a "confused" as opposed to an unequivocal index, and he was well aware of the drift problem inherent in the chain index method.

[79] *Das "Kettensystem"*, das "*zuerst von Lehr und bald darauf von Marshall in Vorschlag gebracht worden ist*" (The "chain system" that "was put forward for consideration first by Lehr and soon thereafter by Marshall", v. Bortkiewicz (1927: 749)).

[80] In this somewhat eccentric concept he primarily saw an attempt to measure utility on the part of Lehr.

[81] Note that this term (unlike the corresponding term in P^{DR}) is a dimensionless ratio of expenditures.

and as this formula is intended to serve as a link $P_{t-1,t}$ in a chain $P_{0t} = P_{01}P_{12}\ldots P_{t-1,t}$ the general formula is given by

$$P^{LE}_{t-1,t} = \frac{\breve{P}_t}{\breve{P}_{t-1}} = \frac{\sum p_{it}q_{it} \Big/ \sum q_{it}\breve{p}_{i,t-1,t}}{\sum p_{i,t-1}q_{i,t-1} \Big/ \sum q_{i,t-1}\breve{p}_{i,t-1,t}} \quad \text{with}$$

$$\breve{p}_{i,t-1,t} = \frac{p_{i,t-1}q_{i,t-1} + p_{it}q_{it}}{q_{i,t-1} + q_{it}}. \tag{13a}$$

The formula (13) looks a bit outlandish[82] and its rightmost variant shows

- that Lehr meant that terms like $S_1 = \sum q_{i1}\breve{p}_{i,01} = \sum q_{i1}(1/g_{i,01})$ and $S_0 = \sum q_{i0}\breve{p}_{i,01}$ represent a sort of total quantity,[83] where (physical) quantities are made commensurable upon dividing by "*Genußeinheiten*", or (equivalently) upon weighting (multiplying) by mid-interval prices $p_{t-1,t}$,[84] so that we obtain with P^{LE}_{0t} a measure of how the price of a pleasure unit has changed.[85], and
- how P^{LE}_{0t} can be viewed as a ratio of a value and a quantity index $P^{LE}_{0t} = V_{0t}/Q^{LE}_{0t}$ and is thereby comparable to $P^{L}_{0t} = V_{0t}/Q^{P}_{0t}$, $P^{P}_{0t} = V_{0t}/Q^{L}_{0t}$, or also $P^{DR}_{0t} = V_{0t}/Q^{D}_{0t}$ so that a ratio of price indices is tantamount to a ratio of quantity indices. In the appendix we will make use of this relationship: $P^{LE}_{0t}/P^{L}_{0t} = Q^{P}_{0t}/Q^{LE}_{0t}$ etc.

In the appendix it also turns out that P^{LE} may in a way be regarded as standing between P^P and P^L. As $\breve{p}_{i,01}$ may be seen as the price in the middle of the interval (0, 1), it is intuitively also plausible that $S_1 = \sum q_1\breve{p}_{01}$ is in a way in the middle of Σq_1p_0 and Σq_1p_1.[86] It follows also that the results of $Q^{LE}_{01} = S_1/S_0$ and $Q^{ME}_{01} = \sum q_1(p_0 + p_1)/\sum q_0(p_0 + p_1)$, the Marshall-Edgeworth quantity index, might be close together.
Introducing price- (λ) and quantity (ω) relatives we get

$$\breve{p}_{i,01} = \frac{p_{i0}(1 + \lambda_{i1}\omega_{i1})}{1 + \omega_{i1}}. \tag{14}$$

It now can easily be seen that if $\lambda_{i1} = 1$ (for all i) $\breve{p}_{i,01} = p_{i0} = p_{i1}$ so that $\sum q_{i1}\breve{p}_{i,01} = \sum q_{i1}p_{i0}$, which Lehr had already noticed,[87] and $P^{LE}_{01} = P^{L}_{01} = P^{P}_{01}$, in which case, however, all these three indices amount to unity. This does not apply, however, for example to P^{DR}_{01}. Hence Lehr's index meets identity (unlike Drobisch's index) but not

[82] Edgeworth called it "cumbrous", which may explain why it was hardly used and never had renowned supporters. Lehr was mentioned by Edgeworth, Walsh and Fisher most rarely in Germany (an exception is Lexis 1886).

[83] For Lehr S_0 and S_1 is the sales value of a number of pleasure units ("*verkaufte Genußeinheiten*" Lehr 1885: 39). Thus the term is expressed in currency units and comparable across all kinds of good. Hence by contrast to Σq_0 and Σq_t in P^{DR} there is no need to express all quantities uniformly in hundredweights (cwt) in P^{LE}. However, S_0 and S_1 may also be regarded as expenditures (values) and Σq_0 and Σq_t are clearly much more understandable.

[84] They may be viewed as "comparable" or "standardized" quantities and therefore much better than Drobisch's simple sums of hundredweights Σq_{it}. Also the Q^{LE} is rightly seen as a sort of quantity index.

[85] "*erhalten wir das Maß, in welchem sich der Preis der Genußeinheit geändert hat*" Lehr (1885: 39).

[86] The same should apply to S_0 in relation to Σq_0p_0 and Σq_0p_1. Note that in this interpretation S_1 (and S_0) is "acting" as an expenditure rather than a "quantity".

[87] Lehr (1885: 40). He also saw that $q_{1t} = q_{2t} = \ldots = q_{nt}$ implies that his index P^{LE} reduces to Dutot's index P^D.

proportionality.[88] $\breve{p}_{i,01} = p_{i0}$ implies $p_{i1} = p_{i0}$ and therefore $\lambda_{i1} = 1$. Note that this result regarding "identity" does not mean that Lehr's index already sufficiently complies with the principle of pure price comparison (i. e. to reflect price changes between 0 and t only) which seems to have been in Laspeyres' thinking the most important criterion a good price index should fulfill. Assuming identity of all prices in 0 and 3 (and also all quantities $q_{i0} = q_{i3}$),[89] multiplying links $P^{LE}_{01}P^{LE}_{12}P^{LE}_{23}$ will in general not result in unity, that is $\bar{P}^{LE}_{03} = P^{LE}_{01}P^{LE}_{12}P^{LE}_{23} \neq 1$ although each link $P_{t\text{-}1,t}$ as such satisfies identity. As is well-known, this not only applies to the chaining of P^{LE}, but also to P^{L} and all sorts of chain indices, which thus violate pure price comparison in the sense of reflecting only the difference between two price vectors $\mathbf{p}_t$ and $\mathbf{p}_0$. Instead chain indices are also affected by prices and quantities of all intermediate periods.

When on the other hand quantities remain constant, that is $\omega_{i1} = 1$ (for all i), $\breve{p}_{i,01} = \frac{1}{2}(p_{i0} + p_{it})$ and Q^{LE}_{0t} reduces to Q^{ME}_{0t} and is unity[90], as is Q^{L}_{0t}, Q^{P}_{0t}, and also Q^{D}_{0t}. Note that $Q^{LE}_{t-1,t} = S_t/S_{t-1}$ compares quantities multiplied by the same prices $\breve{p}_{i,t-1,t}$ and thus can be interpreted as a weighted mean of quantity relatives with weights $q_{i0}\breve{p}_{i,01}/\sum q_{i0}\breve{p}_{i,01}$. Hence unlike $P^{LE}_{t-1,t}$, the quantity index $Q^{LE}_{t-1,t}$ satisfies the mean value property.

According to (11), $\breve{p}_{i,01}$ is a weighted mean $\breve{p}_{i,01} = \alpha p_{i0} + (1-\alpha)p_{it}$ of p_{i0} and p_{i1}, assigning a greater weight $\dfrac{1-\alpha}{\alpha} = \dfrac{q_{i1}}{q_{i0}} > 1$ to the new price p_{i1} then to the old one (p_{i0}) of a commodity when its quantity went up $q_{i1} > q_{i0}$ (that is, $\omega_{i1} > 1$) such that $\sum q_{i1}\breve{p}_{i,01}$ tends to $\sum q_{i1}p_{i1}$ (Paasche) when quantities rise, and to $\sum q_{i1}p_{i0}$(Laspeyres) when they basically go down.

As to the difference between weights $\breve{p}_{i,01}$ and $\breve{p}_{j,01}$ in the sums S_1 and S_0 in Q^{LE} we see that they are related as follows $\dfrac{\breve{p}_{j,01}}{\breve{p}_{i,01}} = \dfrac{g_{i,01}}{g_{j,01}} = \dfrac{p_{j0}}{p_{i0}}\dfrac{1+\omega_j\lambda_j}{1+\omega_i\lambda_i}\dfrac{1+\omega_i}{1+\omega_j}$.[91] Now $\dfrac{q_{j1}\breve{p}_{j,01}}{q_{i1}\breve{p}_{i,01}}$ in Q^{LE} may be compared to q_{j1}/q_{i1} in Q^{D} or to $p_{j0}q_{j1}/\, p_{i0}q_{i1}$ in Q^{L}.

We should refrain from going more into the details of the underlying rationale of P^{LE} and the properties of the index. More importantly, however, it should be noted that Lehr's index – unlike P^{DR}_{0t} – cannot be chained (notwithstanding Lehr had no qualms with multiplying his index numbers) because

$$P^{LE}_{02} = V_{02}\frac{\sum q_{i0}\breve{p}_{i,02}}{\sum q_{i2}\breve{p}_{i,02}} \neq P^{LE}_{01}P^{LE}_{12} = V_{02}\frac{\sum q_{i0}\breve{p}_{i,01}}{\sum q_{i1}\breve{p}_{i,01}}\frac{\sum q_{i1}\breve{p}_{i,12}}{\sum q_{i2}\breve{p}_{i,12}} \tag{15}$$

88 If proportionality ($p_{it} = \lambda p_{i0}$) then also identity (the special case where $\lambda = 1$), but the converse is not true. If identity is violated, so is proportionality. Hence P^{DR} also fails proportionality.

89 With P^{LE} it is not sufficient to multiply only two links under the assumption of price and quantity vectors $\mathbf{p}_0 = \mathbf{p}_2$ *and* $\mathbf{q}_0 = \mathbf{q}_2$ because the *Genusseinheiten* g_i relate two adjacent periods to one another (in a chained Laspeyres index there are no g_i terms, so a chain of two links only suffices to demonstrate that identity may be violated).

90 Put differently: the quantity index of Lehr meets identity (but not proportionality) in the quantities.

91 From this follows: When prices remain constant $\lambda_i = \lambda_j = 1$ good i represents more *Genußeinheiten* than good j when its price is lower ($p_{i0} < p_{j0}$).This is in line with the then widely held opinion that the weight of a price should be inversely proportional to its base period price.

where $P_{12}^{LE} = \frac{\sum p_{i2}q_{i2}}{\sum p_{i1}q_{i1}} \cdot \frac{\sum q_{i1}\breve{p}_{i,12}}{\sum q_{i2}\breve{p}_{i,12}}$, and $\breve{p}_{i,12} = \frac{p_{i1}q_{i1} + p_{i2}q_{i2}}{q_{i1} + q_{i2}}$ (for P_{01}^{LE}(13) applies), and in general we get $P_{0t}^{LE} \neq P_{01}^{LE}P_{12}^{LE}...P_{t-1,t}^{LE}$ as opposed to $P_{0t}^{DR} = P_{01}^{DR}P_{12}^{DR}...P_{t-1,t}^{DR}$.

A final remark to Lehr's ideas on chaining may be added. We could not see that he advanced any noteworthy arguments in order to advocate for chain indices. The only advantage of his approach that he pointed out was the frequently mentioned ease in dealing with the emergence of new goods and disappearance of old goods (or "entry and withdrawal").[92] He argued that abrupt transitions causing extreme discontinuities will be unlikely. And even if there were such abnormal events he considered his method superior to the then possibly widely used strategy to simply cancel outliers in time series and to take averages over longer intervals in time.[93]

In summary his method consisted of:
- taking all observations (in the intermediate periods) into account, not only the endpoints 0 and t of the interval,[94]
- multiplying $P_{t-1,t}$ indices ("links"), to form a chain (that is the chain index method), and
- estimating trends in the time series.

He did not study properties of chain indices and he was not very specific concerning the pros and cons of this method.[95] Such things were not so much in his focus. Instead the greater part of his book is devoted to various least squares estimations of trends in time series of prices (and not to his index formula nor to the rationale of chaining).

8 Some concluding remarks

Given the length of the paper it seems advisable to only very briefly point out some results:

1. To begin with Lehr, it is slightly ironic, and certainly widely unknown, that Germany was one of the first countries, if not the first country, where the idea of chain indices emerged. It is well-known that this country was particularly unhappy with the general move to chain indices in official statistics all over the world in the late 1990s. Chain indices were widely disapproved of as being irreconcilable with pure price comparisons, and were viewed with suspicion.

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[92] "*Tritt nun ein neues Gut ein, … so kann dasselbe einfach in der oben mitgetheilten Formel in Rechnung gestellt werden. Ebenso ist zu verfahren, wenn ein bislang begehrtes Gut fortan … nicht mehr in den Handel gebracht wird*" Lehr (1885: 46). This reads as follows: When a new good emerges … it can simply be accounted for in the above mentioned formula. One may proceed likewise when a hitherto demanded good is henceforth …no longer on the marketplace.

[93] He criticized this method in which he viewed an attempt to detect a sort of trend by excluding extraordinary observations in an otherwise smooth time series. For him the problem was: to exclude the abnormal (e.g. outliers) requires to know what is "normal", and finding out exactly this is the purpose of smoothing (p. 48).

[94] Lehr set great store by taking all price and quantity observations of a time interval into account. Here he vigorously disagreed with Paasche. Also Lehr paid a great deal of attention to the estimation of a linear or exponential trend with the method of least squares. Compared to the index formula, a much greater part of his book is devoted to precisely this task.

[95] He seems to have seen no more (or different) advantages of chaining than other authors of the time also did.

2. It is possibly not a coincidence that all four of the authors presented here only temporarily worked on indices, that they had no idea of the increasing importance this topic would gain in the future and that they underrated the relevance of their index formula. They were occupied with many other research interests and price indices were not central to them. None of them dealt with index numbers for many years or even decades,[96] unlike for example Edgeworth, where around 16 papers on index numbers authored by him are known of, spanning the time period from 1883 to 1925. Furthermore, index problems did not seem to attract many discussants. The situation was fundamentally different in monetary theory where many economists contributed papers and were involved in theoretical disputes. Perhaps misconceptions in this field and the inability to recognize that prices pose intellectually challenging measurement problems could also be responsible for the lack of interest in index numbers.

3. On the other hand, possibly as an after-effect of historicism in economics, much effort was spent on meticulously compiled statistics in laborious monographs covering phenomena of regionally and temporally only rather limited relevance. To give one example only, Paasche extensively studied prices of manorial estates of Prussian knights and other nobles. Paasche might have considered such works, and he made quite a few of this kind, as no less important than his formula. Laspeyres complained (in Laspeyres 1875) that he had to spend some four hours every day only on performing mechanical and dull computations. Much of the work was also devoted to the procurement of statistical data, so there was not much room left for applying one's own index formula, because of the time-consuming preparation of detailed tables. As already mentioned, official statistics of the time did not yet provide statistical data to the extent we are used to today.

4. An astonishing observation for me was that neither Paasche nor Laspeyres were very clear and resolute as far as the specific features of their formulas are concerned. Laspeyres' arguments in favour of q_0 were not very well substantiated (the same applies to Paasche with his choice of q_t). It is not quite clear whether he preferred q_0 to q_t on theoretical grounds, or because q_t might be less readily available than q_0. However, there was an abundance of other statements on methodological issues that may allow us to infer what motivated him to his formula. Surprisingly, he also still adhered to the unweighted Carli index many years after having developed his own formula.

5. In a similar vein, Lehr was most non-committal concerning the justification and effects of the operation of chaining. This is all the more astonishing as he was quite mathematically oriented for his time. In this situation it should have been an interesting exercise for him to do more in the unveiling of properties of his slightly peculiar formula and of chain indices in general.

6. Laspeyres' emphasis on "pure" comparisons had a lasting effect. This was to become distinctive of typical German index theorizing, but also by degrees more of a burden. In a good way it prevented overly "formal" considerations as an end in themselves, but in a bad way it carried on into futile sophistry about the logic of comparability, which characterized German economic statistics in the 1920s and 1930s. One of those ultimately useless topics discussed above is for example the alternative AOR or ROA (also given that many index functions can be written in both ways).

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[96] In the case of Lehr it should be taken into account that he died deplorably early (shortly before his 49th birthday).

Appendix

Relationships between price indices

a) Drobisch, Laspeyres, and Paasche

To show how P^{DR} is related to P^{L} and P^{p} the theorem of Ladislaus von Bortkiewicz on linear indices will be used.[97] We also make use of the equations relating the indices to the value index, viz.$P^{DR}_{0t} = V_{0t}/Q^{D}_{0t}$, $P^{L}_{0t} = V_{0t}/Q^{P}_{0t}$ and $P^{P}_{0t} = V_{0t}/Q^{L}_{0t}$. The theorem then yields the following bias formulas

$$\frac{P^{DR}_{0t}}{P^{L}_{0t}} - 1 = \frac{Q^{P}_{0t}}{Q^{D}_{0t}} - 1 = \frac{\sum \left(\frac{q_t}{q_0} - Q^{D}_{0t}\right)(p_t - \tilde{p}^*_t)w}{Q^{D}_{0t}\tilde{p}^*_t} \tag{16}$$

with weights $w = \frac{q_0}{\sum q_0}$, $Q^{D}_{0t} = \sum \frac{q_t}{q_0} w$, and $\tilde{p}^*_t = \sum p_t w$, and

$$\frac{P^{DR}_{0t}}{P^{P}_{0t}} - 1 = \frac{Q^{L}_{0t}}{Q^{D}_{0t}} - 1 = \frac{\sum \left(\frac{q_t}{q_0} - Q^{D}_{0t}\right)(p_0 - \tilde{p}_0)w}{Q^{D}_{0t}\tilde{p}_0} \quad \text{where } \tilde{p}_0 = \sum p_0 w. \tag{17}$$

The equations are closely related to equations in Diewert and von der Lippe 2010,[98] and they seem to make sense: When changes in quantities correlate negatively with the price level in t we expect the Laspeyres index P^{L} to exceed Drobisch's index P^{DR}, which according to Laspeyres (1871: 307) seems to consistently be the case.[99] For the bias of P^{DR} relative to P^{P}, what matters is the price level of the base period.

b) Drobisch and Lehr

A similar equation with quantity weights $w = q_0/\Sigma q_0$ can be found with Bortkiewicz's theorem for the relationship between $X_1 = P^{DR}$ and $X_0 = P^{LE}$

$$\frac{P^{DR}_{0t}}{P^{LE}_{0t}} - 1 = \frac{Q^{LE}_{0t}}{Q^{D}_{0t}} - 1 = \frac{\sum \left(\frac{q_t}{q_0} - Q^{D}_{0t}\right)(\breve{p}_{01} - \bar{Y})w}{Q^{D}_{0t}\bar{Y}} \tag{18}$$

where $\bar{Y} = Y_0 = \sum \breve{p}_{01} w$ compared to $\tilde{p}^*_t = \sum p_t w$ and $\tilde{p}_0 = \sum p_0 w$ in (16) and (17) respectively. Note that the prices $\breve{p}_{01}$ are quantity weighted averages between prices

[97] See von der Lippe (2007: 198) for this theorem. We follow also the notation with X_0, Y_0, and X_1 used there. I only later realized that v. Bortkiewicz already presented his theorem in a form which I (and many other authors) assumed to be a generalization.

[98] Eq. (17) is equivalent to eq (20) in Diewert and v. d. Lippe, and (16) is basically the same as (29) and (30), where, however, reciprocal quantity relatives, i. e. $r = q_0/q_t$, are studied and all the covariance equations were derived without reference to v. Bortkiewicz. This shows that there is in general more than only one way to describe the relationship between any two linear indices as a function of a covariance. This can be seen here for example with the two equations (19) and (19a), both derived with Bortkiewicz's theorem, or with the fact that we may express $P^{LE}/P^{L} - 1$ on the one hand and $P^{L}/P^{LE} - 1$ (in (16a)) on the other hand using different covariances (the same applies to P^{LE}/P^{P} relative to P^{P}/P^{LE}).

[99] The result also resembles the well-known fact (found by von Bortkiewicz) that $P^{L} > P^{P}$ when price relatives and quantity relatives are negatively correlated. However, weights are then expenditure shares $p_0q_0/\Sigma p_0q_0$ rather than quantity shares $q_0/\Sigma q_0$ as above.

p_0 and p_1, and so in a way are mid-interval prices. The structure of the three formulas (16) through (18) is thus quite similar.

c) Lehr, Laspeyres, and Paasche

It appears desirable to find an expression analogous to (16) with P^{LE} instead of P^{DR}, that is $\frac{P^{LE}_{0t}}{P^{L}_{0t}} - 1 = \frac{Q^{P}_{0t}}{Q^{LE}_{0t}} - 1$, however, one way to construct the formula for this situation according to Bortkiewicz's theorem leads to expenditure weights depending on prices $\breve{p}_{01}$ (and in the case of P^{LE}/P^{P} also to a sort of reciprocal price relatives $p_0/\breve{p}_{01}$) with no plausible interpretation, $\frac{P^{LE}_{0t}}{P^{L}_{0t}} - 1 = \frac{\sum \left(\frac{q_t}{q_0} - Q^{LE}_{0t}\right)\left(\frac{p_1}{\breve{p}_{01}} - \bar{Y}_1\right)\breve{w}}{Q^{LE}_{0t}\bar{Y}}$ with weights

$\breve{w} = \frac{\breve{p}_{01} q_0}{\sum \breve{p}_{01} q_0}$ and $\bar{Y}_1 = \frac{\sum p_1 q_0}{\sum \breve{p}_{01} q_0}$, and

$$\frac{P^{LE}_{01}}{P^{P}_{01}} - 1 = \frac{Q^{L}_{01}}{Q^{LE}_{01}} - 1 = \frac{\sum \left(\frac{q_t}{q_0} - Q^{LE}_{0t}\right)(\breve{p}_{01} - \bar{Y}_2)\breve{w}}{Q^{LE}_{0t}\bar{Y}} \quad \text{where } \bar{Y}_2 = \frac{\sum p_0 q_0}{\sum \breve{p}_{01} q_0}$$

is a kind of reciprocal price index. It appears more reasonable to study the relation $P^{L}/P^{LE} - 1$ (and accordingly $P^{P}/P^{LE} - 1$) instead of $P^{LE}/P^{L} - 1$ (analogous to (16)). This will at least in the case of $P^{P}/P^{LE} - 1$ yield more meaningful weights representing now empirical expenditure shares $p_0q_0/\Sigma\, p_0q_0$, which in turn allows a comparison of the result with the well known formula for the bias $P^{P}/P^{L} - 1$.

$$\frac{P^{L}_{01}}{P^{LE}_{01}} - 1 = \frac{Q^{LE}_{01}}{Q^{P}_{01}} - 1 = \frac{1}{Q^{P}_{01}\bar{Y}} \sum \left(\frac{q_1}{q_0} - Q^{P}_{01}\right)\left(\frac{\breve{p}_{01}}{p_1} - \bar{Y}_3\right)\frac{p_1 q_0}{\sum p_1 q_0} \tag{16a}$$

where $\bar{Y}_3 = \frac{\sum \breve{p}_{01} q_0}{\sum p_1 q_0} = \frac{1}{\bar{Y}_1}$ and

$$\frac{P^{P}_{01}}{P^{LE}_{01}} - 1 = \frac{Q^{LE}_{01}}{Q^{L}_{01}} - 1 = \frac{1}{Q^{L}_{01}\bar{Y}_4} \sum \left(\frac{q_1}{q_0} - Q^{L}_{01}\right)\left(\frac{\breve{p}_{01}}{p_0} - \bar{Y}_4\right)\frac{p_0 q_0}{\sum p_0 q_0} \tag{17a}$$

where $\bar{Y}_4 = \frac{\sum \breve{p}_{01} q_0}{\sum p_0 q_0} = \frac{1}{\bar{Y}_2}$. and this equation can be compared to the well-known equation

$$\frac{P^{P}_{01}}{P^{L}_{01}} - 1 = \frac{1}{Q^{L}_{01}P^{L}_{01}} \sum \left(\frac{q_1}{q_0} - Q^{L}_{01}\right)\left(\frac{p_1}{p_0} - P^{L}_{01}\right)\frac{p_0 q_0}{\sum p_0 q_0} \tag{19}$$

and a formula analogous to (16a) can also be established by the lesser-known equation

$$\frac{P^{L}_{01}}{P^{P}_{01}} - 1 = \frac{1}{Q^{P}_{01}P^{P}_{01}} \sum \left(\frac{q_1}{q_0} - Q^{P}_{01}\right)\left(\frac{p_0}{p_1} - \frac{1}{P^{L}_{01}}\right)\frac{p_0 q_1}{\sum p_0 q_1}, \tag{19a}$$

so that we may compare (17a) to (19), or $\bar{Y}_4$ to P_{01}^L, or (a bit less impressive because of different weights) $\bar{Y}_3 = \frac{\sum \breve{p}_{01} q_0}{\sum p_1 q_0} = \frac{1}{\bar{Y}_1}$ to $\frac{\sum p_0 q_0}{\sum p_1 q_0} = \frac{1}{P_{01}^L}$ in (16a) and (19a).

Moreover it should be borne in mind that the terms $\frac{\breve{p}_{01}}{p_0} = \frac{q_0}{q_0 + q_1} + \frac{q_1}{q_0 + q_1} \cdot \frac{p_1}{p_0}$ in (17a) are simply linear transformations of the price relatives p_1/p_0 and the structure of $\bar{Y}_4$ is similar to that of P_{01}^L. Likewise $\frac{\breve{p}_{01}}{p_1} = \frac{q_1}{q_0 + q_1} + \frac{q_0}{q_0 + q_1} \cdot \left(\frac{p_1}{p_0}\right)^{-1}$ in (16a) may be regarded as linear transformations of *reciprocal* price relatives and the corresponding indices $\bar{Y}_3$ in (16a) and $1/P_{01}^L$ in (19a) as reciprocal price indices.

As is well-known, according to (19) we expect P^L to exceed P^P price relatives and quantity relatives are negatively correlated. In this case linear transformations $\breve{p}_{01}/p_0$ of price relatives will also correlate negatively with quantity relatives, so that we expect $P^P < P^{LE}$ just like $P^P < P^L$. We may also conclude that the negative correlation between p_1/p_0 or $\breve{p}_{01}/p_0$ and the q_1/q_0 amounts to a positive correlation between the reciprocal (transformed) price relatives and the quantity relatives so that we will have $P^L > P^{LE}$ just like $P^L > P^P$. Hence there are good reasons to assume that Lehr's price index (P^{LE}) lies within the bounds of Paasche (P^P) and Laspeyres (P^L), such that $P^P < P^{LE} < P^L$ (or, less likely, $P^P > P^{LE} > P^L$).

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