# A GENERAL FORMULAE

In this appendix A to the article Internal validation of temporal disaggregation: A cloud chamber approach I outline the derivation of $\phi^*$ as a function of $m$ and $\rho$ given the aggregation problem described in the main text. First, I write (6) for general $m$ and then show how $S_0$ and $S_1$ result. The value of $\phi^*$ then results by applying (9).

The starting point is the notion that any $z_{h,t}$ can be given as

$$z_{h,t} = \rho^n z_{h,t-n} + \rho^{n-1}\epsilon_{h,t-n+1} + \rho^{n-2}\epsilon_{h,t-n+2} + \cdots + \epsilon_{h,t} \tag{A.1}$$

which implies for temporal aggregation over $m$ periods,

$$\begin{aligned}
z_{h,t} + z_{h,t-1} + \cdots + z_{h,t-m+1} &= \rho^m z_{h,t-m} + \rho^{m-1}\epsilon_{h,t-m+1} + \rho^{m-2}\epsilon_{h,t-m+2} + \cdots + \epsilon_{h,t} \\
&+ \rho^m z_{h,t-m-1} + \rho^{m-1}\epsilon_{h,t-m} + \rho^{m-2}\epsilon_{h,t-m+1} + \cdots + \epsilon_{h,t-1} \\
&\vdots \\
&+ \rho^m z_{h,t-2m+1} + \rho^{m-1}\epsilon_{h,t-2m+2} + \rho^{m-2}\epsilon_{h,t-2m+3} + \dots \\
&\quad +\epsilon_{h,t-m+1} \\
&= \rho^m (z_{h,t} + z_{h,t-1} + \cdots + z_{h,t-m+1}) + u_{l,\tau}.
\end{aligned}$$

$$z_{l,\tau} = \rho^m z_{l,\tau-1} + u_{l,\tau} \tag{A.2}$$

where the error term $u_{l,\tau}$ is the sum of the elements of the $(m \times m)$ matrix $\Phi_\tau$:

$$\Phi_\tau = \left[\epsilon_t \ \epsilon_{t-1} \ \dots \ \epsilon_{t-m+1}\right]' \otimes \left[(\rho L)^{m-1} \ (\rho L)^{m-2} \ \dots \ (\rho L)^0\right] \tag{A.3}$$

which makes use of the lag operator, $L$, with $L^i x_t = x_{t-i}$. It is instructive to expand $\Phi_\tau$:

$$\Phi_\tau = \begin{pmatrix}
\rho^{m-1}\epsilon_{h,t-m+1} & \rho^{m-2}\epsilon_{h,t-m+2} & \rho^{m-3}\epsilon_{h,t-m+3} & \cdots & \rho^0\epsilon_{h,t} \\
\rho^{m-1}\epsilon_{h,t-m} & \rho^{m-2}\epsilon_{h,t-m+1} & \rho^{m-3}\epsilon_{h,t-m+2} & \cdots & \rho^0\epsilon_{h,t-1} \\
\rho^{m-1}\epsilon_{h,t-m-1} & \rho^{m-2}\epsilon_{h,t-m} & \rho^{m-3}\epsilon_{h,t-m+1} & \cdots & \rho^0\epsilon_{h,t-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\rho^{m-1}\epsilon_{h,t-2m+1} & \rho^{m-2}\epsilon_{h,t-2m+2} & \rho^{m-3}\epsilon_{h,t-2m+3} & \cdots & \rho^0\epsilon_{h,t-m+1}
\end{pmatrix} \tag{A.4}$$

This matrix has an interesting structure. In particular, the innovations with identical time subscripts are to be found along the diagonals. Thus, the variance of $u_{l,\tau}$ is the sum of the squared sums of the diagonal elements. At the same time the power to which $\rho$ is raised is the same in each column. Therefore, every secondary diagonal can be regarded a truncated version of the main diagonal with respect to the power coefficients. The following auxiliary matrices and operator are useful in finding handy expressions. Let me use the operator *diag* which stacks the main diagonal of a symmetric matrix into a vector. Hence,

$$
\begin{aligned}
\Psi &\equiv 1_{m\times 1}\left[\rho^{m-1}\ \rho^{m-2}\ \dots\ \rho^{0}\right] \\
diag(\Psi) &= \left[\rho^{m-1}\rho^{m-2}\rho^{m-3}\cdots\rho^{0}\right]' \\
&= \psi' \\
H &\equiv \begin{pmatrix}
1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 1 & 1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 1 & 1 & \cdots & 0 & 0 \\
0 & 0 & \ddots & 1 & 1 & 1 & \ddots & 1 & 0 \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 1
\end{pmatrix}
\end{aligned}
$$

where $diag(\Psi)$ and $H$ have dimensions $(m \times 1)$ and $(m \times 2m-1)$ respectively, and $1_{m\times 1}$ is a $(m \times 1)$ vector of ones. Notice that $H$ is essentially a matrix of $m$ rows of a $m$ dimensional column vector of ones within a $(m \times 2m-1)$ matrix of zeros where in each successive row the vector of ones is shifted one column to the right. The product $\psi H$ now conveniently collects the $2m-1$ sums of the diagonal elements of $\Phi_\tau$ in a $(1 \times 2m-1)$ vector omitting for the sake of simplicity the innovation terms. The variance of $u_{l,\tau}$ can now be obtained as

$$
\begin{aligned}
S_0 &\equiv \psi H H' \psi' \\
E(u_{l,\tau}u_{l,\tau}) &= \sigma_h^2 S_0
\end{aligned}
$$

which makes use of the *i.i.d.* property of the $\epsilon_t$.

For deriving $S_1$, decompose $H = (h_1, 1_{m\times 1}, h_2)$ where $h_1$ and $h_2$ are $(m \times m-1)$ matrices collecting the sums of the diagonal elements below and above the main diagonal respectively.

Consider now $\Phi_{\tau-1} = L^m \Phi_\tau$ whose sum of elements define $u_{l,\tau-1}$. The value of $S_1$ is linear in the covariance between $u_{l,\tau}$ and $u_{l,\tau-1}$. Therefore, we need to multiply the sums of the elements above the main diagonal of the matrix $\Phi_{\tau-1}$ with the sums of the elements below the main diagonal of the matrix $\Phi_\tau$ diagonal by diagonal. With the aid of $h_1$ and $h_2$ one can write

$$\begin{aligned} S_1 &= \psi h_1 h_2' \psi' \\ E(u_{l,\tau} u_{l,\tau-1}) &= \sigma_h^2 S_1. \end{aligned}$$

The discussion of the identification issues easily generalises to the case for arbitrary $m$ by noting that again $|\frac{S_0}{2S_1}| > \sqrt{\frac{S_0^2}{4S_1^2} - 1}$ if $\rho \neq 0$ and that $\frac{1}{2\rho}\frac{S_0}{\frac{S_1}{\rho}}$ in general implies identical signs for $\rho$ and $\phi^*$. Therefore, in the case of even $m$ one might contemplate choosing the invertible MA coefficient out of the two possible for identifying the disaggregate model. As argued before, identification is ensured for uneven $m$.